The Fundamental Flaw in Gödel's Incompleteness Theorem "On Formally Undecidable Propositions of Principia Mathematica and Related Systems"

(1) Note

This paper may be considered to be a in depth demonstration of the flaw in Gödel's Incompleteness Theorem as given in the author's novel, "The Shackles of Conviction", published by *james R meyer* Publishing. Please see the author's website for details (www.jamesrmeyer.com).

(2) Abstract

This paper identifies the fundamental error inherent in Gödel's Incompleteness theorem. The error is generated by the ambiguity of the language of Gödel's Proposition V, a proposition for which Gödel declined to furnish a detailed proof. The error arises from a confusion of the meta-language and the languages to which it refers, a confusion which is exacerbated by the failure of Gödel to clarify the principal assertions involved in his suggested proof outline, where there is a reliance on intuition rather than logical transparency.

The result of this vagueness of presentation is that there is no clear delineation of the meta-language and the sub-languages involved, with the result that an equivalence is asserted between an expression of the meta-language and a sub-language, an equivalence which is logically untenable.

It is shown here that this means that the self-reference generated by Proposition VI in Gödel's proof relies on this erroneous intuitive assumption, and hence the self-reference of that Proposition is logically untenable, and there is no logical basis for Gödel's result.

This paper uses straightforward logic and does not rely on any philosophical or semantical arguments.

(3) Contents

(4) Preface

Gödel's proof is perhaps one of the most celebrated proofs in the entire history of mathematics, and is considered to be a tour-de-force in the generation of a result by logical derivation. It may come as a surprise to discover that Gödel's result is only achievable by confusion and ambiguity inherent in the presentation of his proof. Gödel's result relies on hidden assumptions that under rigorous examination are seen to have no logical basis.

A common interpretation of Gödel's result is "There is a formula of the formal system, which is not provable by the formal system, but it is true" (as in Proposition VI of Gödel's paper).

However, a logical interpretation of Gödel's result does not require any reference to truth, and may be stated as: *"If we accept the assumptions and the rules of logical deduction of the language of Gödel's proof, then in every formal system that includes number theory, there is a formula of number theory which is not provable by the rules and axioms of that formal system, but is provable by a combination of those rules and axioms of that formal system together with the assumptions and rules of logical deduction of the language of Gödel's proof."*

The assertion that the formal expression referred to by Gödel's proof, which is not provable by the formal system, is nonetheless true, introduces a spurious notion of truth, a notion that is vague and undefined. Some observers have suggested that the formula is not proven in the language of Gödel's theorem, but merely demonstrated to be 'true'. This is of course, playing semantical games with the term 'true' while at the same time refusing to define it. Gödel's paper itself refers to the statement as 'decidable' – "*The proposition which is undecidable in the [formal] system turns out to be decided by meta-mathematical considerations."*

The term 'decidable' is no different in principle to the term 'provable', and it follows that there must be some '*considerations'* used by the language of Gödel's proof that generate the decision that are not available to the formal system.[1]

This of course demonstrates the paradoxical nature of Gödel's result – that for any formal system there is a formula which is unprovable by the formal system, but is provable by the language of Gödel's proof. If Gödel's proof is correct, doesn't that indicate that Gödel's proof can never be stated in a formal language? But on the other hand, if Gödel's proof is a logically coherent argument from specifiable first principles, then ultimately, should it not be possible to transform this logical argument to a precisely defined formal language? It is quite remarkable that there have been so many misguided and illogical attempts to explain away this fundamentally paradoxical result, rather than attempting to seek out the root source of the paradox.

The result of Gödel's proof rests on the assumptions and rules of the language of the proof. If it is supposed that Gödel's proof is valid, the question arises, "Is Gödel's result obtainable in all possible consistent proof languages?" It also gives rise to the question, "Are there consistent proof languages where Gödel's result is not obtainable?" If that is the case (and by Gödel's proof that must be the case for all formal languages) then if Gödel's result is valid, that must mean that the language of Gödel's proof is in some sense 'stronger', because it can prove information about the formal system that the other language can not prove.

If Gödel's proof were to be in some sense universally 'true', then for any consistent proof language, it must either be the case that it is not 'strong enough' to generate Gödel's result or that it is 'strong enough' to generate Gödel's result. This means that if a consistent proof language is not 'strong enough' to generate Gödel's result, then that language must lack at least one rule or assumption that a proof language capable of generating Gödel's result possesses, or that it includes at least one additional rule or assumption that prevents it from generating Gödel's result. And if Gödel's proof is correct, then that applies to all formal

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[1] There are even those who reject any notion that the 'unprovable' formula given by Gödel's theorem can in any way be considered to be true or provable. Among such curious notions is that Gödel's proof only asserts a conditional: *"If the formal system is consistent, then there is a formula of the formal system, which is not provable by the formal system, but it is true."* Such arguments ignore the obvious fact that every proved expression is dependent upon the assumptions used in its generation; such conditionals could be pedantically applied to the result of every proof, and there is no obvious reason why this conditional should be treated any differently. In any case, the acceptance of Gödel's proof must also include the assumption that Gödel's proof language itself is consistent. Since Gödel's proof is asserted to apply for any formal system that includes arithmetic, if it is assumed that there exists at least one consistent formal system that includes arithmetic, one can dispense with the above conditional. It would be completely absurd to assume that Gödel's proof language (which itself includes arithmetic) itself is consistent, while at the same time asserting that one cannot assume that there exists at least one consistent formal system that includes arithmetic.

languages, which indicates a fundamental difference between all formal languages and a language that can generate Gödel's result, where that fundamental difference is at least one rule or assumption. Since Gödel's proof only uses a finite amount of language, we would expect that there can only be a finite number of such rules or assumptions in Gödel's proof language, and it would be surprising indeed if these rules or assumptions could not be discovered.

We note that Gödel's proof is a proof by rejection of a contradiction. At one step in the proof, one of the possible implications of an assertion is rejected because it would lead to a contradiction. That rejection of an implication assumes that none of the assumptions, whether explicit or implicit, used to reach that point of the proof are logically incoherent, and so dismisses the possibility that the language of the proof is inherently contradictory. However, if all the assumptions that are made in arriving at that point in the proof are not clearly defined, then there is a very real possibility that the rejection of one of the possible implications is invalid, since the root source of the contradiction may be a prior erroneous assumption.

Gödel's result depends on his Proposition V, and the proof of this Proposition V relies on certain assumptions. As with any mathematical proof, whether Gödel's proof can be considered to be 'proved' depends on whether the assumptions used can be accepted as valid by conventional standards of logic. The assumptions that Gödel makes in order to achieve his result are here shown to fly in the face of all standards of commonly accepted logic. Far from making the language 'stronger', Gödel's result demonstrates the pitfalls involved when there is ambiguity of language, and reliance on intuition. There are indeed proof languages which are consistent but where Gödel's result is not achievable. In such languages there is no room for the assumptions that are inherent in Gödel's proof, and no basis for the conclusions of Gödel's proof.

The location of the logical flaw in Gödel's proof lies within Gödel's Proposition V, a proposition for which Gödel dismisses any need for a comprehensive proof on the basis that *"... it offers no difficulties of principle and is somewhat involved"*, and merely offers an outline proof. The result of this vagueness of presentation is that there is no clear delineation of the meta-language and the sub-languages involved, with the result that an equivalence is asserted between an expression of the meta-language and a sub-language, an equivalence which is logically untenable.

The argument presented in this paper relates to Part 2 of Gödel's paper, and not the prefatory argument presented in Part 1 of Gödel's paper which is sometimes incorrectly referred to as though it were the proof proper. Contrary to some commonly expressed opinions, this prefatory argument was not used by Gödel as the basis for the main proof, although both the prefatory argument and the main argument share the same confusion of meta-language and sub-language. A brief synopsis of the obvious flaw in that argument is presented in Appendix 1: Gödel's Prefatory Argument.

The author notes that there are many so-called proofs which are asserted to be 'versions' of Gödel's proof. A detailed analysis of every current so-called 'version' of Gödel's proof is beyond the scope of this paper, but the author notes the following. Many of these so-called 'versions' of Gödel's proof are based on some version of Gödel's prefatory argument and so can be similarly dispatched. There are also so-called 'versions' that are based on such concepts as the Berry Paradox and algorithmic complexity, and which are also subject to the same flaw as Gödel's proof, although the methods used are somewhat different. Moreover, many of the so-called 'versions' of Gödel's proof do not give a resultant expression of which it can be claimed that is *'not provable by the formal system, but it is true/provable'*, and hence cannot be considered to be a 'version' of Gödel's proof. Typical of these so-called 'versions' are those that are based on Turing's halting argument.[a] While there are certain similarities, assertions that these are essentially the same as Gödel's proof fail to understand the distinction; Turing's argument does not give a formula which is supposedly true but not provable. Turing himself noted that while *'... conclusions are reached which are superficially similar to those of Gödel'*, he also pointed out, *'It should perhaps be remarked what I shall prove is quite different from the well-known results of Gödel'*. [2]

2 It should perhaps be noted that there are also some deficiencies in Turing's argument, but it is beyond the scope of this paper to address them here.

(5) Introduction

(5.1) Symbols and terms used in this paper

0	≡	zero	$\forall$	≡	for all
f	≡	the successor of	$\Rightarrow$	≡	implies
$\neg$	≡	not	$\leftrightarrow$	≡	correspondence
$\vee$	≡	or	$\wedge$	≡	and
$\exists$	≡	there exists	$\equiv$	≡	equivalence

Various relations are referred to and explained in the main text. The following is only a summary of the principal relations referred to

Φ(*X*)	The function that describes what is commonly referred to as the Gödel numbering system, see p.9
Ψ(*X*)	A function that gives a specific natural number of the meta-language for every symbol of the formal system, see p.9
***R*, *R*(*x*)**	A relation of natural numbers
$\boldsymbol{FRM}^{\mathbf{F}}$	A formula of the formal system **F**
$\boldsymbol{PRF}^{\mathbf{F}}$	A proof scheme of the formal system **F** that is a proof of some formula
$\boldsymbol{x\ Proof^{F}\ y}$	The symbol combination ***x*** is a proof in the system F of the symbol combination ***y***
$\boldsymbol{T}^{\mathbf{A}}$	A mapping function
0, *f*0, *ff*0, *fff*0, …,	Symbol combinations for natural numbers
***Fr*(*x*, *y*)**	***x*** is a free variable in the symbol sequence ***y***

↲ used to indicate for some expressions that the line continues on the next line, and that the lines are not separate expressions.

The following are relations that are defined in Gödel's paper. The version of Gödel's paper referred to here is the English translation of Gödel's paper by B. Meltzer.[b] There is also a translation by Martin Hirzel (27 Nov 2000) and the corresponding terms are indicated below.

***Subst*(*a*, *v*, *b*)**	Gödel's meta-mathematical notion of substitution, see p.12 ($\boldsymbol{subst\ a\begin{pmatrix} v \\ \hline b \end{pmatrix}}$ in Hirzel's translation)
***Z*(*v*)**	Gödel's relation 17 (***number*(*v*)** in Hirzel's translation)
***Sb*(*x*, *v*, *w*)**	Gödel's relation 31 (***subst*(*x*, *v*, *w*)** in Hirzel's translation)
x B v	Gödel's relation 45 (***proofFor*(*x*, *v*)** in Hirzel's translation)
***Bew*(*x*)**	Gödel's relation 46 (***provable*(*x*)** in Hirzel's translation)

Note that the definition of the terms *'recursive'* and ***'ω–consistent'*** are not here defined, since these terms are defined in Gödel's paper, and their precise definition is immaterial to the argument here presented.

(5.2) The Gödel numbering system – a brief overview

In a formal system, the basic symbols of the system are simply placed one after another in a particular order to create a formula of the system. In his paper, Gödel defines a relationship between symbol sequences of a formal system and numbers. This is referred to as a one-to-one correspondence. The system that Gödel defined to establish this relationship between a sequence of symbols of the formal system and a number is basically as follows:

First, Gödel defines a relationship between each symbol of the formal system that is not a variable to an associated number as below.

$\mathbf{0} \leftrightarrow \mathbf{1}$	$\neg \leftrightarrow \mathbf{5}$	$\forall \leftrightarrow \mathbf{9}$	$) \leftrightarrow \mathbf{13}$
$f \leftrightarrow \mathbf{3}$	$\vee \leftrightarrow \mathbf{7}$	$(\leftrightarrow \mathbf{11}$	

Each variable of the first type of the formal system (that is, with the domain of individuals of the formal system, that is, entities of the form **0, *f*0, *ff*0, *fff*0, …**) is defined as corresponding to a prime number greater than 13, i.e., 17*,* 19*,* 23*, …* . For convenience we shall refer to these variables of the formal system as $\boldsymbol{v_1, v_2, v_3, \ldots}$ [3]

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[3] The details of the symbols for variables of the second type or higher are irrelevant to the argument presented in this paper and are not considered here.

The above relationship is a one-to-one correspondence, that is, a bijective function – we designate this as **Ψ(*t*)** [4]. The value of **Ψ(*t*)** is the number that corresponds to the symbol given by ***t***, that is, it returns one of the values 1, 3, 5, 7, 9, 11, 13, 17, 19, 23, … etc.

Gödel defines the relationship of a natural number to a symbol sequence of the formal system as follows:
The symbol at the n^{th} position is represented by the n^{th} prime number to the power of the number corresponding to that symbol. This method of assigning a number to sequence of formal symbols is also a one-to-one correspondence, i.e., a bijective function. This function is designated as **Φ(*X*)**, where ***X*** is a sequence of formal symbols. The formal symbol sequence ***X*** may be of any finite length, and the function **Φ(*X*)** references by the function **Ψ(*t*)** every symbol of the symbol sequence ***X***.

(5.3) Meta-language and sub-language

Gödel's paper involves the notion of a meta-language. In his paper the proof language is considered to be a meta-language and the formal system to be referenced by that meta-language. The formal system is considered to be simply a collection of specific symbols: **0, *f*, ¬, ∀, ∨,), (**, together with the symbols for variables. The symbols for variables of the formal system are not symbols for variables in the meta-language, but are only specific values in the meta-language – that is, the rules of syntax that apply to variables of the meta-language do not apply to the symbols for variables of the formal language. For the moment we ignore the implications where the same symbols are used for relational operators in the meta-language and the formal system (for example, "∀" might be used as the universal quantifier symbol in both the formal language and the meta-language).

Notes:

- In the rest of this paper, we shall refer to a language where the variables of that language are not symbols for variables in the meta-language as a sub-language to the meta-language.

- In the rest of this paper, capital symbols are used to represent variables of the proof language, to make it easier to distinguish symbols that are variables of the proof language and symbols that are variables of a sub-language such as the formal system.

- In this paper, for convenience, the proof language in which Gödel's Proposition V is stated is called the language **PV.**

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[4] While Gödel does not explicitly define a function with the designation Ψ(*t*), his definition of the numbering system implicitly defines this function.

(6) Layout of this Paper

It has to be remarked at this point that the persistent misunderstanding of Gödel's proof lies largely in the vagueness in which certain concepts are expressed in his paper. These include the following:

- the concept of *'number-theoretic relations'* is central to the proof, yet it is a concept for which no definition at all is given in Gödel's paper. Presumably Gödel assumed that it is intuitively obvious what constitutes a *'number-theoretic relation'*.
- no proof is given for the key proposition, Proposition V, and the suggested outline of a proof does little more than refer to recursion, a concept that is not even required to give Gödel's result.
- there is a confusion of the terms truth and provability.
- there is confusion resulting from the use of a meta-language that uses the same symbols for relational operators as sub-languages.
- the key proposition, Proposition V, is a proposition where there is no clear delineation of meta-language and sub-language.

The confusion generated by such vagueness creates very real barriers to understanding. These however are not insurmountable, and this paper aims to clarify the concepts so that a full understanding of Gödel's theorem and the flaws therein may be achieved.

Section (7) is an overview of the mapping of sub-languages by a meta-language, a concept vaguely referred to by Gödel's proof, but one for which Gödel gives no detailed consideration.

Section (8) give a detailed overview of Gödel's Proposition V, a proposition for which Gödel declined to furnish a detailed proof, stating that it presents *"no difficulties in principle"* and that it is "*intuitionistically unobjectionable"* (his footnote 45a) and merely indicates the outline of a proof. The aim of this overview is to clarify the argument given by Gödel's sketchy outline of how a proof might proceed, and by so doing set the concepts involved on a firm logical footing, inasmuch as that can be achieved, bearing in mind the flaw therein.

Section (9) gives a much simpler proof of Gödel's principal result that is a *'true but unprovable formula'*, (as given by Gödel's Propositions VI) that is obtained without any reference to recursion or **ω**–consistency, but which uses the principles of the outline of Gödel's proof. Again, the inherent flaw is ignored in this section.

Section (10) demonstrates the flaw in Gödel's proof.

Section (11) is a brief summary, followed by the conclusion - Section (12)

(7) Sub-languages and Mapping Functions

Before examining Gödel's Proposition V in depth, it is instructive to consider a simple mapping of one formal system ***A*** to another formal system ***B***, since we shall be using certain terms later in the text that are explained here.

In simple terms, such a mapping might be described by:

(7.1.1) $\forall \boldsymbol{FRM}^{\mathbf{A}}$,

(7.1.2) $\exists \boldsymbol{PRF}^{\mathbf{A}}, \{\boldsymbol{PRF}^{\mathbf{A}}\ \boldsymbol{Proof}^{\mathbf{A}}\ \boldsymbol{FRM}^{\mathbf{A}}\} \Rightarrow \exists \boldsymbol{FRM}^{\mathbf{B}}, \exists \boldsymbol{PRF}^{\mathbf{B}}, \{\boldsymbol{PRF}^{\mathbf{B}}\ \boldsymbol{Proof}^{\mathbf{B}}\ \boldsymbol{FRM}^{\mathbf{B}}\}$

where $\boldsymbol{PRF}^{\mathbf{A}}$ and $\boldsymbol{PRF}^{\mathbf{B}}$, $\boldsymbol{FRM}^{\mathbf{A}}$ and $\boldsymbol{FRM}^{\mathbf{B}}$ are variables of the meta-language – their domain is symbol combinations of the formal languages ***A*** and ***B*** respectively (which may be proof schemes or formulas), and
$\boldsymbol{Proof}^{\mathbf{A}}$ is a relation, where $\boldsymbol{d}\ \boldsymbol{Proof}^{\mathbf{A}}\ \boldsymbol{e}$ means that the symbol combination ***d*** is a proof in the language ***A*** of the symbol combination ***e***, and similarly for $\boldsymbol{Proof}^{\mathbf{B}}$.

The expression may be interpreted as:
*For every formula of the formal system **A**, if there exists a formal proof scheme in the system **A** which is the proof of that formula, that implies that there exists a corresponding formula in the formal system **B**, for which there exists a formal proof scheme in the formal system **B**.*

The proposition is a proposition in a meta-language where both the formal system ***A*** and the formal system ***B*** are sub-languages to the meta-language, and where the variables of these formal languages are not variables of the meta-language. The expressions (7.1.1)-(7.1.2) are expressions of the meta-language regarding entities of the sub-languages ***A*** and ***B***.

The basis for the assertion of the proposition is that there exists a mapping function $\boldsymbol{T}^{\mathbf{A\text{-}B}}$ in the language of the proposition[5], and this mapping function asserts that for a given formula of the system ***A***, there exists a formula of the system ***B***. This gives:

(7.1.3) $\forall \boldsymbol{FRM}^{\mathbf{A}}, \exists \boldsymbol{FRM}^{\mathbf{B}}, \{\boldsymbol{FRM}^{\mathbf{B}} = \boldsymbol{T}^{\mathbf{A\text{-}B}}(\boldsymbol{FRM}^{\mathbf{A}})$

(7.1.4) $\exists \boldsymbol{PRF}^{\mathbf{A}}, \{\boldsymbol{PRF}^{\mathbf{A}}\ \boldsymbol{Proof}^{\mathbf{A}}\ \boldsymbol{FRM}^{\mathbf{A}}\} \Rightarrow \exists [\boldsymbol{T}^{\mathbf{A\text{-}B}}(\boldsymbol{PRF}^{\mathbf{A}})], \{ [\boldsymbol{T}^{\mathbf{A\text{-}B}}(\boldsymbol{PRF}^{\mathbf{A}})]\ \boldsymbol{Proof}^{\mathbf{B}}\ [\boldsymbol{T}^{\mathbf{A\text{-}B}}(\boldsymbol{FRM}^{\mathbf{A}})]\}$

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5 Note that the assertion of the existence of a mapping function $\boldsymbol{T}^{\mathbf{A\text{-}B}}$ in the language of the proposition is an assertion that can only be made in a language that is a meta-language to the language of the proposition.

Or, designating $\boldsymbol{T}^{\text{A-B}}(\boldsymbol{PRF}^{\text{A}})$ by $\boldsymbol{PRF}^{\text{B}}$, $\boldsymbol{T}^{\text{A-B}}(\boldsymbol{FRM}^{\text{A}})$ by $\boldsymbol{FRM}^{\text{B}}$ this gives:

(7.1.5) $\forall \boldsymbol{FRM}^{\text{A}}, \exists \boldsymbol{FRM}^{\text{B}}, \{\boldsymbol{FRM}^{\text{B}} = \boldsymbol{T}^{\text{A-B}}(\boldsymbol{FRM}^{\text{A}})$

(7.1.6) $\exists \boldsymbol{PRF}^{\text{A}}, \{\boldsymbol{PRF}^{\text{A}}\ \boldsymbol{Proof}^{\text{A}}\ \boldsymbol{FRM}^{\text{A}}\} \Rightarrow \exists \boldsymbol{PRF}^{\text{B}}, \{\boldsymbol{PRF}^{\text{B}}\ \boldsymbol{Proof}^{\text{B}}\ \boldsymbol{FRM}^{\text{B}}\}$

which is similar to the expression given by (7.1.1) & (7.1.2) above.

Gödel uses the concept of ***Subst*** in his proof: "By ***Subst*(*a*, *v*, *b*)** (where ***a*** stands for a formula, ***v*** a variable and ***b*** a sign of the same type as ***v***) we understand the formula derived from ***a***, when we replace ***v*** in it, wherever it is free, by ***b***." Applying this to the above gives:

(7.1.7) $\forall \boldsymbol{FRM}^{\text{A}}, \forall \boldsymbol{W}^{\text{A}}, \forall \boldsymbol{X}^{\text{A}}, \exists \boldsymbol{FRM}^{\text{B}}, \{\boldsymbol{FRM}^{\text{B}} = \boldsymbol{T}^{\text{A-B}}(\boldsymbol{FRM}^{\text{A}})\}, \exists \boldsymbol{W}^{\text{B}}, \{\boldsymbol{W}^{\text{B}} = \boldsymbol{T}^{\text{A-B}}(\boldsymbol{W}^{\text{A}})\},$
$\exists \boldsymbol{X}^{\text{B}}, \{\boldsymbol{X}^{\text{B}} = \boldsymbol{T}^{\text{A-B}}(\boldsymbol{X}^{\text{A}})\},$

(7.1.8) $\exists \boldsymbol{PRF}^{\text{A}}, \{\boldsymbol{PRF}^{\text{A}}\ \boldsymbol{Proof}^{\text{A}}\ \boldsymbol{Subst}(\boldsymbol{FRM}^{\text{A}}, \boldsymbol{W}^{\text{A}}, \boldsymbol{X}^{\text{A}})\}$

(7.1.9)
$\Rightarrow \exists[\boldsymbol{T}^{\text{A-B}}(\boldsymbol{PRF}^{\text{A}})], \{[\boldsymbol{T}^{\text{A-B}}(\boldsymbol{PRF}^{\text{A}})]\ \boldsymbol{Proof}^{\text{B}}\ \boldsymbol{Subst}[\boldsymbol{T}^{\text{A-B}}(\boldsymbol{FRM}^{\text{A}}, \boldsymbol{T}^{\text{A-B}}(\boldsymbol{W}^{\text{A}}), \boldsymbol{T}^{\text{A-B}}(\boldsymbol{X}^{\text{A}})]\}$

(7.1.10) $\Rightarrow \exists \boldsymbol{PRF}^{\text{B}}, \{\boldsymbol{PRF}^{\text{B}}\ \boldsymbol{Proof}^{\text{B}}\ \boldsymbol{Subst}(\boldsymbol{FRM}^{\text{B}}, \boldsymbol{W}^{\text{B}}, \boldsymbol{X}^{\text{B}})\}$

(7.1.11) where the expression ***Subst*** is an expression of the meta-language, and

(7.1.12) $\boldsymbol{W}^{\text{A}}$, $\boldsymbol{W}^{\text{B}}$, $\boldsymbol{X}^{\text{A}}$ and $\boldsymbol{X}^{\text{B}}$ are variables of the meta-language, where

$\boldsymbol{W}^{\text{A}}$ is a variable whose domain is variable symbols of the formal system ***A***,

$\boldsymbol{W}^{\text{B}}$ is a variable whose domain is variable symbols of the formal system ***B***,

$\boldsymbol{X}^{\text{A}}$ is a variable whose domain is number symbols of the formal system ***A***,

$\boldsymbol{X}^{\text{B}}$ is a variable whose domain is number symbols of the formal system ***B***,

and where the mapping function $\boldsymbol{T}^{\text{A-B}}$ applies to give $\boldsymbol{T}^{\text{A-B}}(\boldsymbol{PRF}^{\text{A}}) = \boldsymbol{PRF}^{\text{B}}$

Now this expression is somewhat cumbersome, which is the reason such expressions are commonly reduced to a simpler form, such as:

(7.1.13) $\forall \boldsymbol{FRM}^{\text{A}}, \forall \boldsymbol{W}^{\text{A}}, \forall \boldsymbol{X}^{\text{A}},$

(7.1.14) $\exists \boldsymbol{PRF}^{\text{A}}, \{\boldsymbol{PRF}^{\text{A}}\ \boldsymbol{Proof}^{\text{A}}\ \boldsymbol{Subst}(\boldsymbol{FRM}^{\text{A}}, \boldsymbol{W}^{\text{A}}, \boldsymbol{X}^{\text{A}})\} \Rightarrow \exists \boldsymbol{PRF}^{\text{B}}, \{\boldsymbol{PRF}^{\text{B}}\ \boldsymbol{Proof}^{\text{B}}\ \boldsymbol{Subst}$
$(\boldsymbol{FRM}^{\text{B}}, \boldsymbol{W}^{\text{B}}, \boldsymbol{X}^{\text{B}})\}$

where it is implied that the mapping function $\boldsymbol{T}^{\text{A-B}}$ applies to give

$\boldsymbol{T}^{\text{A-B}}(\boldsymbol{PRF}^{\text{A}}) = \boldsymbol{PRF}^{\text{B}}$, $\boldsymbol{T}^{\text{A-B}}(\boldsymbol{FRM}^{\text{A}}) = \boldsymbol{FRM}^{\text{B}}$, $\boldsymbol{T}^{\text{A-B}}(\boldsymbol{X}^{\text{A}}) = \boldsymbol{X}^{\text{B}}$, $\boldsymbol{T}^{\text{A-B}}(\boldsymbol{W}^{\text{A}}) = \boldsymbol{W}^{\text{B}}$

This expression will suffice provided that the underlying implicit assumptions are not ignored.

(8) Detailed Overview of Proposition V

This detailed overview attempts to clarify various vaguely expressed notions arising from Gödel's outline proof of Proposition V, and thereby attempts to create a quasi-rigorous proof of Proposition V. In this overview, we ignore as far as possible the fundamental flaws inherent in the argument.

(8.1) The Outline Proof of Proposition V

For the purposes of simplification, in this section of this paper we will deal with Gödel's Proposition V

- for relations of only one free variable,
- without consideration of the negation of the *'number-theoretic relation'*, and
- without interpretative descriptions

since the argument here presented does not rely on these aspects of the proposition.

The assertion of Proposition V is thus given as:

(8.1.1) For any recursive 'number-theoretic relation' $\boldsymbol{R(x)}$, for all $\boldsymbol{x}$:
there exist numbers $\boldsymbol{r}$ and $\boldsymbol{u}$, where $\boldsymbol{p}^{\mathbf{u}}$ is a factor of $\boldsymbol{r}$, where $\boldsymbol{p}$ is a prime number, and where $\boldsymbol{u}$ is any prime number greater than 13, and

(8.1.2) $\boldsymbol{R(x) \Rightarrow Bew\{Sb[r, u, Z(x)]\}}$

The implied proof of Proposition V can be considered to consist of four principal assertions:

Assertion I the expression *"there exists an expression A in the formal system that is a proof of the formal system formula B where its free variable has been substituted by some number"* has a corresponding definable *'number-theoretic relation'*.

Assertion II for any recursive *'number-theoretic relation'*, then for any specific values substituted for the free variables of the relation, it is decidable by finite means whether that relation or its negation is *'true'*.

Assertion III for every recursive *'number-theoretic relation'* $\boldsymbol{R}$, there is a corresponding formal system formula $\boldsymbol{FRM}^{\mathbf{F}}$. The formal system formula will have the same number of free variables as the *'number-theoretic relation'*.

Assertion IV if a recursive *'number-theoretic relation'* $\boldsymbol{R}$ is *'true'*(decidable), there exists a formal proof for the corresponding formal system formula $\boldsymbol{FRM}^{\mathbf{F}}$.

(8.2) Assertion I of the proof of Proposition V:

We give here Assertion I of Proposition V:

(8.2.1) $\forall \boldsymbol{FRM}^{\mathbf{F}}, \forall \boldsymbol{W}^{\mathbf{F}}, \forall \boldsymbol{X},$

(8.2.2) $\exists \boldsymbol{PRF}^{\mathbf{F}}, [\boldsymbol{PRF}^{\mathbf{F}}\ \boldsymbol{Proof}^{\mathbf{F}}\ \boldsymbol{Subst}(\boldsymbol{FRM}^{\mathbf{F}}, \boldsymbol{W}^{\mathbf{F}}, \boldsymbol{X})]$

(8.2.3) $\Rightarrow \exists \boldsymbol{\Phi}(\boldsymbol{PRF}^{\mathbf{F}}), \{\boldsymbol{\Phi}(\boldsymbol{PRF}^{\mathbf{F}})\ \boldsymbol{B}\ \boldsymbol{Sb}[\boldsymbol{\Phi}(\boldsymbol{FRM}^{\mathbf{F}}), \boldsymbol{\Psi}(\boldsymbol{W}^{\mathbf{F}}), \boldsymbol{\Phi}(\boldsymbol{X})]\}$

(8.2.4) $\Rightarrow \exists \boldsymbol{Y}, \{\boldsymbol{Y}\ \boldsymbol{B}\ \boldsymbol{Sb}[\boldsymbol{T}, \boldsymbol{W}^{\mathbf{R}}, \boldsymbol{\Phi}(\boldsymbol{X})]\}$

(8.2.5) $\Rightarrow \boldsymbol{Bew}\{\boldsymbol{Sb}[\boldsymbol{T}, \boldsymbol{W}^{\mathbf{R}}, \boldsymbol{\Phi}(\boldsymbol{X})]\}$

where

- $\boldsymbol{a}\ \boldsymbol{Proof}^{\mathbf{F}}\ \boldsymbol{b}$ means ***a*** is a combination of symbols of the formal system that is a complete proof scheme of the formal system for the formula ***b***, and
- $\boldsymbol{PRF}^{\mathbf{F}}$ and $\boldsymbol{FRM}^{\mathbf{F}}$ are variables of the language **PV** – the domain of these variables is symbol combinations of the formal system,
- $\boldsymbol{W}^{\mathbf{F}}$ is a variable of the language **PV**, the domain of which is symbols that are variables of the formal system,
- ***X***, ***Y*** and ***T*** are all variables of the language **PV** of natural numbers, with the domain of expressions of the language **PV** for natural numbers, and

- by the mapping functions **Φ** and **Ψ**, we have $\boldsymbol{\Phi}(\boldsymbol{FRM}^{\mathbf{F}}) = \boldsymbol{T}, \boldsymbol{\Psi}(\boldsymbol{W}^{\mathbf{F}}) = \boldsymbol{W}^{\mathbf{R}}, \boldsymbol{\Phi}(\boldsymbol{PRF}^{\mathbf{F}}) = \boldsymbol{Y}$
- in step (8.2.4), ***Y B A*** is Gödel's relation 45, "***Y*** is a proof of the formula ***X***". The step from (8.2.4) to (8.2.5) follows from the definition given by Gödel's relation 46, which is $\boldsymbol{Bew}(\boldsymbol{A}) \equiv \exists \boldsymbol{Y}, (\boldsymbol{Y}\ \boldsymbol{B}\ \boldsymbol{A})$.

For the specific value $\boldsymbol{v}^{\mathbf{F}}$ of the variable $\boldsymbol{W}^{\mathbf{F}}$, the assertion is that there is a specific corresponding value **17** under the function **Ψ**,and this gives:

(8.2.6) $\forall \boldsymbol{FRM}^{\mathbf{F}}, \forall \boldsymbol{X},$

(8.2.7) $\exists \boldsymbol{PRF}^{\mathbf{F}}\ [\boldsymbol{PRF}^{\mathbf{F}}\ \boldsymbol{Proof}^{\mathbf{F}}\ \boldsymbol{Subst}(\boldsymbol{FRM}^{\mathbf{F}}, \boldsymbol{v}^{\mathbf{F}}, \boldsymbol{X})]$

(8.2.8) $\Rightarrow \exists \boldsymbol{\Phi}(\boldsymbol{PRF}^{\mathbf{F}})\{\boldsymbol{\Phi}(\boldsymbol{PRF}^{\mathbf{F}})\ \boldsymbol{B}\ \boldsymbol{Sb}[\boldsymbol{\Phi}(\boldsymbol{FRM}^{\mathbf{F}}), \boldsymbol{\Psi}(\boldsymbol{v}^{\mathbf{F}}), \boldsymbol{\Phi}(\boldsymbol{X})]\}$

(8.2.9) $\Rightarrow \exists \boldsymbol{Y}\{\boldsymbol{Y}\ \boldsymbol{B}\ \boldsymbol{Sb}[\boldsymbol{T}, \mathbf{17}, \boldsymbol{\Phi}(\boldsymbol{X})]\}$

(8.2.10) $\Rightarrow \boldsymbol{Bew}\{\boldsymbol{Sb}[\boldsymbol{T}, \mathbf{17}, \boldsymbol{\Phi}(\boldsymbol{X})]\}$

where $\mathbf{17} = \boldsymbol{\Psi}(\boldsymbol{v}^{\mathbf{F}})$. $\boldsymbol{v}^{\mathbf{F}}$ is not a variable of the proof language, since it is not subject to a variable quantifier, and neither is it a free variable in the expression (otherwise the expression would not be a proposition - we note that all the expressions (8.2.6) – (8.2.10) are expressions of the language **PV**).

(8.2.11) The function **Φ(*X*)** will give a value that is a natural number in the language **PV** for any given ***X***. A further assertion is that, since the values of ***X*** that occur in the function **Φ(*X*)** in the above expression are only symbol combinations for natural numbers, that is, of the form **0, *f*0, *ff*0, *fff*0, …,** the function ***Z*(*X*)** (Gödel's relation 17) gives the same value as the function **Φ(*X*)** for all such values of the variable ***X***.

Applying this assertion, $\mathbf{\Phi(\mathit{X}) \equiv \mathit{Z}(\mathit{X})}$ to the above gives for a specific variable $\boldsymbol{v}^{\mathbf{F}}$ of the formal system:

(8.2.12) $\forall \boldsymbol{FRM}, \forall \boldsymbol{X},$

(8.2.13) $\exists \boldsymbol{PRF}^{\mathbf{F}}\ [\boldsymbol{PRF}^{\mathbf{F}}\ \boldsymbol{Proof}^{\mathbf{F}}\ \boldsymbol{Subst}(\boldsymbol{FRM}^{\mathbf{F}}, \boldsymbol{v}^{\mathbf{F}}, \boldsymbol{X})]$

(8.2.14) $\Rightarrow \exists \boldsymbol{\Phi}(\boldsymbol{PRF}^{\mathbf{F}})\{\boldsymbol{\Phi}(\boldsymbol{PRF}^{\mathbf{F}})\ \boldsymbol{B}\ \boldsymbol{Sb}[\boldsymbol{\Phi}(\boldsymbol{FRM}^{\mathbf{F}}), \boldsymbol{\Psi}(\boldsymbol{v}^{\mathbf{F}}), \boldsymbol{\Phi}(\boldsymbol{X})]\}$

(8.2.15) $\Rightarrow \exists \boldsymbol{\Phi}(\boldsymbol{PRF}^{\mathbf{F}})\{\boldsymbol{\Phi}(\boldsymbol{PRF}^{\mathbf{F}})\ \boldsymbol{B}\ \boldsymbol{Sb}[\boldsymbol{\Phi}(\boldsymbol{FRM}^{\mathbf{F}}), \boldsymbol{\Psi}(\boldsymbol{v}^{\mathbf{F}}), \boldsymbol{Z}(\boldsymbol{X})]\}$

(8.2.16) $\Rightarrow \exists \boldsymbol{Y}\{\boldsymbol{Y}\ \boldsymbol{B}\ \boldsymbol{Sb}[\boldsymbol{T}, \mathbf{17}, \boldsymbol{Z}(\boldsymbol{X})]\}$

(8.2.17) $\Rightarrow \boldsymbol{Bew}\{\boldsymbol{Sb}[\boldsymbol{T}, \mathbf{17}, \boldsymbol{Z}(\boldsymbol{X})]\}$

where the expressions (8.2.12) – (8.2.17) are expressions of the language **PV**.

(8.3) Assertion III and Assertion IV of the Proof of Proposition V

Assertion III of the proof of Proposition V asserts that for every recursive *'number-theoretic relation'*, there is a corresponding formal system formula. This applies, therefore, to recursive *'number-theoretic relations'* with one free variable. Assertion IV asserts that if that recursive *'number-theoretic relation'* ***R*** is *'true'* for some value of that free variable, then there is a formal proof for the corresponding formal system formula $\boldsymbol{FRM}^{\mathbf{F}}$ when the free variable of that formula is substituted by that same value. This results in the proposition:

(8.3.1) $\forall \boldsymbol{R}, \forall \boldsymbol{X}, \exists \boldsymbol{FRM}^{\mathbf{F}}$

(8.3.2) $\boldsymbol{R}(\boldsymbol{X}) \Rightarrow \exists \boldsymbol{PRF}^{\mathbf{F}}\ [\boldsymbol{PRF}^{\mathbf{F}}\ \boldsymbol{Proof}^{\mathbf{F}}\ \boldsymbol{Subst}(\boldsymbol{FRM}^{\mathbf{F}}, \boldsymbol{v}^{\mathbf{F}}, \boldsymbol{X})]$

While it might be asserted that a correspondence exists between ***R*** and $\boldsymbol{FRM}^{\mathbf{F}}$, there is no explanation for how the part of the expression that is $\exists \boldsymbol{PRF}^{\mathbf{F}}$ **[…]** is generated, or how it follows either from the expression ***R*(*X*).** It simply appears to arise from nowhere at all, and as such is completely unacceptable. However, it is instructive to see how the *'intuitive'* assertion of Gödel's Proposition V arises. Gödel's footnote 39 states that *"Proposition V naturally is based on the fact that for any recursive relation **R**, it is decidable, for every n-tuple of numbers, from the axioms of the system P, whether the relation **R** holds or not"*. [6]

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[6] See also Section Appendix 2: Provability and Truth.

This is a rather roundabout way of asserting that there exists a proof either of the relation $\boldsymbol{R}$ or its negation for any specific values of its free variables, which would give:

(8.3.3) $\forall \boldsymbol{R}, \forall \boldsymbol{X}, \exists \boldsymbol{FRM}^{\mathbf{F}}$

(8.3.4) $\exists \boldsymbol{PRF}^{\mathbf{R}}[\boldsymbol{PRF}^{\mathbf{R}}\ \boldsymbol{Proof}^{\mathbf{R}}\ (\boldsymbol{R}(\boldsymbol{X}))] \Rightarrow \exists \boldsymbol{PRF}^{\mathbf{F}}\ [\boldsymbol{PRF}^{\mathbf{F}}\ \boldsymbol{Proof}^{\mathbf{F}}\ \boldsymbol{S}\mathbf{ub}\boldsymbol{s}\mathbf{t}(\boldsymbol{FRM}^{\mathbf{F}}, \mathbf{v}^{\mathbf{F}}, \boldsymbol{X})]$

where

- $\boldsymbol{a}\ \boldsymbol{Proof}^{\mathbf{R}}\ \boldsymbol{b}$ means that $\boldsymbol{a}$ is a combination of symbols of the language of the relation $\boldsymbol{R}$ that is a complete proof scheme of the combination of symbols $\boldsymbol{b}$ of the language of the relation $\boldsymbol{R}$, and
- $\boldsymbol{PRF}^{\mathbf{R}}$ is a variable of the language **PV**, the domain of which is symbol combinations of the language of the relation $\boldsymbol{R}$.

Now, while we now have an explanation for the assertion that there is a valid generation of $\exists \boldsymbol{PRF}^{\mathbf{F}}\ [\boldsymbol{PRF}^{\mathbf{F}} \ldots]$, there is clearly an implied assertion of a mapping function such that $\boldsymbol{PRF}^{\mathbf{F}} = \boldsymbol{T}^{\mathbf{R\text{-}F}}[\boldsymbol{PRF}^{\mathbf{R}}]$ and this has not been explicitly included in the above expression. Besides that omission, there is however still no satisfactory explanation for how the expression $\boldsymbol{Subst}(\boldsymbol{FRM}^{\mathbf{F}}, \boldsymbol{v}^{\mathbf{F}}, \boldsymbol{X})$ arises from the expression $\boldsymbol{R}(\boldsymbol{X})$. While the symbol $\boldsymbol{X}$ in the expression $\boldsymbol{R}(\boldsymbol{X})$ can map to the symbol $\boldsymbol{X}$ in the expression $\boldsymbol{Subst}(\boldsymbol{FRM}^{\mathbf{F}}, \boldsymbol{v}^{\mathbf{F}}, \boldsymbol{X})$, there remains no definition of how a mapping from $\boldsymbol{R}$ to the entities $\boldsymbol{FRM}^{\mathbf{F}}$ and $\boldsymbol{v}^{\mathbf{F}}$ is to be achieved.

Now suppose that we have a system that is the same as the formal system, but differs in just one symbol; for the sake of argument we say that a variable, say $\boldsymbol{v}_1$, of Gödel's formal system is replaced by the symbol $\boldsymbol{y}$ in this system. By the definition of the formal system of Gödel's proof, this system is not the formal system that is the subject of Gödel's proof. But surely it must be a system where at least some of its valid expressions are *'number-theoretic relations'*? And in that case, that begs the question which has no satisfactory answer: why would it be asserted, as in (8.3.4) above, for some expression $\boldsymbol{R}$ concerning numbers in that system that:

(8.3.5) $\exists \boldsymbol{PRF}^{\mathbf{R}}[\boldsymbol{PRF}^{\mathbf{R}}\ \boldsymbol{Proof}^{\mathbf{R}}\ (\boldsymbol{R}(\boldsymbol{X}))] \Rightarrow \exists \boldsymbol{PRF}^{\mathbf{F}}\ [\boldsymbol{PRF}^{\mathbf{F}}\ \boldsymbol{Proof}^{\mathbf{F}}\ \boldsymbol{Subst}(\boldsymbol{FRM}^{\mathbf{F}}, \boldsymbol{v}^{\mathbf{F}}, \boldsymbol{X})]$

rather than:

(8.3.6) $\exists \boldsymbol{PRF}^{\mathbf{R}}[\boldsymbol{PRF}^{\mathbf{R}}\ \boldsymbol{Proof}^{\mathbf{R}}\ (\boldsymbol{Subst}[\boldsymbol{R}, \boldsymbol{y}, \boldsymbol{X}])] \Rightarrow \exists \boldsymbol{PRF}^{\mathbf{F}}\ [\boldsymbol{PRF}^{\mathbf{F}}\ \boldsymbol{Proof}^{\mathbf{F}}\ \boldsymbol{Subst}(\boldsymbol{FRM}^{\mathbf{F}}, \boldsymbol{v}^{\mathbf{F}}, \boldsymbol{X})]$?

since the two formal systems are almost identical?

(8.4) A Clarification of 'Higher-Order Logic'

Proposition V is a proposition with two explicitly stated quantified variables (that is, governed by a universal quantifier "***For all***"), a variable ***R*(*x*)** whose domain is a set of *'number-theoretic relations'* and a variable ***x*** whose domain is the set of natural numbers,

Such expressions are commonly referred to as being expressions of higher-order or second-order logic, where as well as normal variables, there are relation variables. However, this is simply an appellation which of itself does not confer on the expression any logical validity. Gödel's Proposition V is in a commonly used format, and is of the form:

(8.4.1) $\forall \boldsymbol{R(x)}, \forall \boldsymbol{x}, \boldsymbol{Expression(R, x)}$

where

- ***R*(*x*)** is a variable whose domain is those *'number-theoretic relations'* that have ***x*** as a free variable, and
- ***x*** is a variable whose domain is the set of natural numbers

As it stands, such an expression is ambiguous. On the one hand, the variables of the language of the relation ***R*** (of which ***x*** is one) are treated as symbols that are not variables of the language of the expression – it is implied by the expression (8.4.1) that: "***R*** *is an expression that according to the rules of the language of the relation* ***R*** *contains only one free variable, and that free variable is the symbol* ***x***". Clearly, ***x*** cannot be a variable where it occurs in "∀***R*(*x*)**", since:

(8.4.2) ***x*** cannot be a free variable where it occurs in "∀***R*(*x*)**", since the entire expression (8.4.1) is a proposition, and

(8.4.3) ***x*** cannot be a bound variable where it occurs in "∀***R*(*x*)**", since it is not subject to a quantifier within the expression "∀***R*(*x*)**"

On the other hand, the expression ***Expression*(*R*, *x*)** appears to be an expression of the language of the entire expression where the symbol ***x*** is a variable of the language of the expression. This means that in the entire expression (8.4.1), the symbol ***x*** appears to be both a variable and not a variable of the expression at the same time.

Hence the expression as it stands is not a clear proposition, since the determination of which expressions are to be implied by this expression depends on the interpretation of the expression. Similarly, the determination of which expressions are to be expressions that prove the proposition depends on the interpretation of those expressions and the proposition itself. This is particularly so in the determination of whether a symbol is to be perceived as a variable of such expressions. This scenario is clearly not acceptable for logical analysis.

It should perhaps be noted that Gödel certainly did not define any rules for a clear interpretation of his Proposition V and any expressions purported to be a proof thereof, although he did assume that the language of his proof could not result in a contradiction.

In any case, the above expression (8.4.1) violates a fundamental principle of logical analysis. That fundamental principle is that all occurrences of a symbol that is a variable in an expression may be simply replaced by any other symbol for a variable, provided that that symbol is not already in the expression. If we try to apply that principle to the naïve expression $\forall \boldsymbol{R(x)}, \forall \boldsymbol{x}, \boldsymbol{Expression(R, x)}$, replacing every occurrence of the symbol $\boldsymbol{x}$ where it is a variable by the symbol $\boldsymbol{y}$ this gives:

(8.4.4) $\quad \forall \boldsymbol{R(x)}, \forall \boldsymbol{y}, \boldsymbol{Expression(R, y)}$

since in the ambiguous language used, $\boldsymbol{x}$ is not a variable where it occurs in "$\forall \boldsymbol{R(x)}$" whereas the desired result is:

(8.4.5) $\quad \forall \boldsymbol{R(y)}, \forall \boldsymbol{y}, \boldsymbol{Expression(R, y)}$

since the implied assertion in all such expressions is that the assertion of the expression also applies to all relations of one free variable, and not only relations with the actual variable symbol used ($\boldsymbol{x}$ in this example).

Various attempts to circumvent this problem have been attempted where the problem is conveniently swept under the carpet and then ignored, with the result that the fundamental properties of propositions and variables are obscured. There is no obvious reason why such misguided attempts have been made, since an expression that unambiguously expresses the desired concept without defining the domain of the variable $\boldsymbol{R}$ in terms of an entity that is elsewhere defined as a variable is readily achieved. The required expressions can readily be made while at the same time retaining quantifiers on the variables of the expression. Instead of asserting that the domain of the relation $\boldsymbol{R}$ is all relations with one free variable $\boldsymbol{x}$, all that needs to be asserted is that the domain of $\boldsymbol{R}$ is all relations, with an included condition on the entity $\boldsymbol{x}$, such as "*If **R** is an expression that according to the rules of the language of the relation **R** contains only one free variable, and that free variable is the symbol **x**, then …*".

This gives, instead of the expression (8.4.1):

(8.4.6) $\quad \forall \boldsymbol{R}, \{\boldsymbol{C(R, x)} \Rightarrow \boldsymbol{Expression(R, x)}\}$

where

- $\boldsymbol{R}$ is a variable whose domain is *'number-theoretic relations'*, and
- $\boldsymbol{C(R, x)}$ is the condition that $\boldsymbol{x}$ is a free variable in the relation $\boldsymbol{R}$, and there are no other free variables in $\boldsymbol{R}$.

We note furthermore that it is not necessary to stipulate that the variable $\boldsymbol{R}$ in the above expressions has the domain of *'number-theoretic relations'*, since this can be included in the condition, viz:

(8.4.7) $\forall \boldsymbol{Y}, \{\boldsymbol{C}(\boldsymbol{Y}, \boldsymbol{x}) \Rightarrow \boldsymbol{Expression}(\boldsymbol{Y}, \boldsymbol{x})\}$

where $\boldsymbol{C}(\boldsymbol{Y}, \boldsymbol{x})$ is the condition that $\boldsymbol{Y}$ is a *'number-theoretic relation'*, $\boldsymbol{x}$ is a free variable in the relation $\boldsymbol{Y}$, and there are no other free variables in $\boldsymbol{Y}$.

In the following for convenience, we refer to the $\boldsymbol{R}$ in the expressions as having the domain of *'number-theoretic relations'*, though in principle, the requirement that $\boldsymbol{R}$ be a *'number-theoretic relation'* could be included in the condition $\boldsymbol{C}$.

In principle we assume that there is no difficulty in precisely defining a condition such as $\boldsymbol{C}(\boldsymbol{R}, \boldsymbol{x})$; in any case, such a definition is required to define the domain of $\boldsymbol{R}$ in Gödel's naïve expression (8.4.1). When the expression is stated in this format, it is quite evident that $\boldsymbol{x}$ cannot be a variable anywhere in the expression since:

(8.4.8) $\boldsymbol{x}$ cannot be a free variable in $\boldsymbol{C}(\boldsymbol{R}, \boldsymbol{x})$, since the expression is a proposition, and

(8.4.9) $\boldsymbol{x}$ cannot be a bound variable in $\boldsymbol{C}(\boldsymbol{R}, \boldsymbol{x})$, since it is not subject to a quantifier.

It follows that in $\boldsymbol{Expression}(\boldsymbol{R}, \boldsymbol{x})$ any reference to $\boldsymbol{x}$ is to $\boldsymbol{x}$ as an entity that is not a variable of the entire expression, since the expression is now expressed in a clearly defined fashion.

The expression (8.4.6) implies the expression given when any valid specific value of $\boldsymbol{R}$ is substituted for $\boldsymbol{R}$, for example:

(8.4.10) $\boldsymbol{C}(\boldsymbol{R}_1, \boldsymbol{x}) \Rightarrow \boldsymbol{Expression}(\boldsymbol{R}_1, \boldsymbol{x})$

where $\boldsymbol{R}_1$ is any valid value of the variable $\boldsymbol{R}$.

Furthermore, there may be expressions with other quantified variables (that are not variables in the relation $\boldsymbol{R}$), for example,

(8.4.11) $\forall \boldsymbol{R}, \forall \boldsymbol{Y}, \{\boldsymbol{C}(\boldsymbol{R}, \boldsymbol{x}) \Rightarrow \boldsymbol{Expression}(\boldsymbol{R}, \boldsymbol{Y}, \boldsymbol{x})\}$

There is also an implied assertion in all expressions such as $\forall \boldsymbol{R}(\boldsymbol{x}), \forall \boldsymbol{x}, \boldsymbol{Expression}(\boldsymbol{R}, \boldsymbol{x})$ that the expression where the symbol $\boldsymbol{x}$ is replaced by some other variable symbol[7] is also a valid expression. We may readily apply this principle to the expression (8.4.6) which is $\forall \boldsymbol{R}, \{\boldsymbol{C}(\boldsymbol{R}, \boldsymbol{x}) \Rightarrow \boldsymbol{Expression}(\boldsymbol{R}, \boldsymbol{x})\}$. Since $\boldsymbol{x}$ is not a variable in the expression, we may generalise the expression, by the introduction of another quantified variable, which is readily assimilated into the expression (8.4.6), giving:

(8.4.12) $\forall \boldsymbol{R}, \forall \boldsymbol{W}, \{\boldsymbol{C}(\boldsymbol{R}, \boldsymbol{W}) \Rightarrow \boldsymbol{Expression}(\boldsymbol{R}, \boldsymbol{W})\}$

where

$\boldsymbol{R}$ is a variable whose domain is *'number-theoretic relations'*,

$\boldsymbol{W}$ is a variable whose domain is symbols for variables in *'number-theoretic relations'*, and

$\boldsymbol{C}(\boldsymbol{R}, \boldsymbol{W})$ is the condition that $\boldsymbol{W}$ is a free variable in the relation $\boldsymbol{R}$.

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[7] Where the symbol is a variable in the same language as the language in which $\boldsymbol{x}$ is a variable symbol.

The above deals with the case where an expression explicitly references the symbol of a variable such as $\boldsymbol{x}$ in the relation subject to the quantifier, as in $\forall \boldsymbol{R(x)}$. However, simply removing that explicit reference to a symbol such as $\boldsymbol{x}$ does not ensure an absence of ambiguity in the expression.

As a case in point, we may consider an example of a higher-order formula – an expression referred to as the axiom of induction, which is commonly stated as:

(8.4.13) $\forall \boldsymbol{R}, \{(\boldsymbol{R}(0) \wedge \forall \boldsymbol{x}[\boldsymbol{R}(\boldsymbol{x}) \Rightarrow \boldsymbol{R}(\boldsymbol{x}+1)] \Rightarrow \forall \boldsymbol{x}\boldsymbol{R}(\boldsymbol{x})\}$
which is interpreted as: *"For every relation R, if R is true of* 0*, and if for every n, R(n) implies R(n+1), then R is true of every n*."

However, there is an implicit assertion that ***R***, where referenced by "$\forall \boldsymbol{R}$", is a relation with only one free variable, but this is not explicit in the expression as given. If we express this explicitly by applying a condition on ***R*** as *"If **R** is a relation with one free variable, then …"* we get:

(8.4.14) $\forall \boldsymbol{R}, \{\boldsymbol{C}(\boldsymbol{R}) \Rightarrow [(\boldsymbol{R}(0) \wedge \forall \boldsymbol{x}[\boldsymbol{R}(\boldsymbol{x}) \Rightarrow \boldsymbol{R}(\boldsymbol{x}+1)] \Rightarrow \forall \boldsymbol{x}\boldsymbol{R}(\boldsymbol{x})]\}$
where $\boldsymbol{C}(\boldsymbol{R})$ is the condition that there is one free variable in the relation ***R***. That condition will express, *"There exists a **W**, such that **W** is a free variable in **R**, and there does not exist a **Z**, such that **Z** ≠ **W**, and **Z** is a free variable in **R**"*, which gives:
(8.4.15) $\boldsymbol{C}(\boldsymbol{R}) \equiv \exists \boldsymbol{W}, \boldsymbol{Fr}(\boldsymbol{R}, \boldsymbol{W})\ \&\ \neg\exists \boldsymbol{Z},[(\boldsymbol{Z} \neq \boldsymbol{W}) \wedge \boldsymbol{Fr}(\boldsymbol{R}, \boldsymbol{Z})]$
where $\boldsymbol{Fr}(\boldsymbol{R}, \boldsymbol{A})$ means that ***A*** is a free variable in ***R***. Again, since the condition is implicit in the original expression (8.4.13), if the original expression is a valid logical expression, that condition must in principle be expressible.

Clearly, if the entire expression is to be a valid expression, this condition on ***R*** must be explicitly expressible as part of the expression, since the condition on ***R*** is implicit in the expression. When stated explicitly, it is evident that there must be some variable such as ***W*** in order to express the condition. The domain of this variable ***W*** is symbols that may be variables in the relation ***R***. This also means that if we have an expression such as $\forall \boldsymbol{W}, \boldsymbol{Expression}(\boldsymbol{W})$ which is a valid proposition (and hence has no free variable) in the same language as that of the expression (8.4.14), that expression implies that the statement ***Expression*(*c*)**, where ***c*** is some member of the domain of ***W***, is a proposition with no free variables. It follows that ***c*** cannot be a variable in the language of the expression (8.4.14), since it is neither subject to a quantifier nor is it a free variable (otherwise the expression ***Expression*(*c*)** would not be a proposition).

This means that in the expression:

(8.4.16) $\forall \boldsymbol{R}, \{\boldsymbol{C}(\boldsymbol{R}) \Rightarrow [(\boldsymbol{R}(0) \wedge \forall \boldsymbol{x}[\boldsymbol{R}(\boldsymbol{x}) \Rightarrow \boldsymbol{R}(\boldsymbol{x}+1)] \Rightarrow \forall \boldsymbol{x}\boldsymbol{R}(\boldsymbol{x})]\}$, which is equivalent to

(8.4.17) $\forall \boldsymbol{R}, \{(\exists \boldsymbol{W}, \boldsymbol{Fr}(\boldsymbol{R}, \boldsymbol{W}) \,\&\, \neg\exists \boldsymbol{Z},[(\boldsymbol{Z} \neq \boldsymbol{W}) \wedge \boldsymbol{Fr}(\boldsymbol{R}, \boldsymbol{Z})]) \Rightarrow$ ↲

$[(\boldsymbol{R}(0) \wedge \forall \boldsymbol{x}[\boldsymbol{R}(\boldsymbol{x}) \Rightarrow \boldsymbol{R}(\boldsymbol{x}+1)] \Rightarrow \forall \boldsymbol{x}\boldsymbol{R}(\boldsymbol{x})]\}$

that either:

(8.4.18) the language of the entire expression is unambiguous and the symbol ***x*** cannot be a variable in the expression, since it is a member of the domain of the variable ***W***, or

(8.4.19) the entire expression is in an ambiguous language. However, the rules for this ambiguous language are not clearly defined, and the notion that this is a suitable basis for a language for logical analysis is absurd. Nor has been shown that such a language is in any way necessary for logical analysis – that is, it has not been shown that there is some deficiency in a language that is clearly defined. [8]

To state the expression in a language that is unambiguous, if it is acknowledged that the language of the relation ***R*** is a sub-language (which we call the language **S**) to the language of the entire expression, this gives:

(8.4.20) $\forall \boldsymbol{R}, \forall \boldsymbol{W}, \{\boldsymbol{C}(\boldsymbol{R}, \boldsymbol{W}) \Rightarrow \boldsymbol{Provable}_1^{\mathrm{S}}[\boldsymbol{R}(0)] \wedge \boldsymbol{Provable}_1^{\mathrm{S}}[\forall^{\mathrm{S}}\boldsymbol{W}(\boldsymbol{R}(\boldsymbol{W}) \Rightarrow^{\mathrm{S}} \boldsymbol{R}(\boldsymbol{W}+1))]$ ↲

$\Rightarrow \boldsymbol{IC}^{\mathrm{S}}[\forall^{\mathrm{S}}\boldsymbol{W}(\boldsymbol{R}(\boldsymbol{W})]\}$

where

- $\forall^{\mathrm{S}}$ is the symbol for ***"For all"*** in the language **S**,
- $\Rightarrow^{\mathrm{S}}$ is the symbol for ***"implies"*** in the language **S**,
- ***W*** is a variable whose domain is symbols for variables of the language **S**,
- ***C*(*R*, *W*)** is the condition that ***W*** is a free variable in the relation ***R***, and there are no other free variables in ***R***,
- $\boldsymbol{Provable}_1^{\mathrm{S}}(\boldsymbol{a})$ means that ***a*** is an expression that is provable according to the axioms and rules of the language **S** as already defined.
- we use the terminology ***R*(0)** to represent ***Subst*(*R*, *W*, 0)**, ***R*(*W*+1)** to represent ***Subst*(*R*, *W*, *W* +1)**, and ***R*(*W*)** to represent ***R***. These are all expressions in the meta-language, which represent expressions of the sub-language **S** upon substitution of the meta-language variable ***W*** by a symbol that is a variable in the sub-language **S.**

We note that the above expression may be considered to be a rule of inference, rather than an axiom, so that $\boldsymbol{IC}^{\mathrm{S}}(\boldsymbol{a})$ means that ***a*** is an formula that is an immediate consequence of the other expressions ***R*(0)** and $\forall^{\mathrm{S}}\boldsymbol{W}(\boldsymbol{R}(\boldsymbol{W}) \Rightarrow^{\mathrm{S}} \boldsymbol{R}(\boldsymbol{W}+1))$ of the language **S**, where ***W*** is some variable symbol of the language **S**. In general, if a proof exists for a formula, then there will

[8] Of course, in one sense, it could be said that this is what Gödel's proof asserts. However, an assertion by a vaguely defined ambiguous language that that language itself is required to 'prove' that that language itself is somehow superior to a language that is not ambiguous is quite clearly absurd.

be a series of formulas, each of which is either an axiom or is an immediate consequence of previous formulas. Alternatively, we could state that $\boldsymbol{IC}^{S}\{\boldsymbol{a}, \boldsymbol{R}(\mathbf{0}), \forall^{S}\boldsymbol{W}(\boldsymbol{R}(\boldsymbol{W}) \Rightarrow^{S} \boldsymbol{R}(\boldsymbol{W}+\mathbf{1}))\}$ means that $\boldsymbol{a}$ is an immediate consequence of the expressions $\boldsymbol{R}(\mathbf{0})$ and $\forall^{S}\boldsymbol{W}(\boldsymbol{R}(\boldsymbol{W}) \Rightarrow^{S} \boldsymbol{R}(\boldsymbol{W}+\mathbf{1}))$, provided $\boldsymbol{R}(\mathbf{0})$ and $\forall^{S}\boldsymbol{W}(\boldsymbol{R}(\boldsymbol{W}) \Rightarrow^{S} \boldsymbol{R}(\boldsymbol{W}+\mathbf{1}))$ are either themselves immediate consequences or axioms.

Again, we use different symbols for relational operators to avoid having symbols whose syntactical interactions are ambiguous. For a given variable symbol $\boldsymbol{x}$ of the language **S**, the expression (8.4.20) implies:

(8.4.21) $$\forall \boldsymbol{R}, \{\boldsymbol{C}(\boldsymbol{R}, \boldsymbol{x}) \Rightarrow \boldsymbol{Provable}_1{}^{S}[\boldsymbol{R}(\mathbf{0})] \wedge \boldsymbol{Provable}_1{}^{S}[\forall^{S}\boldsymbol{x}(\boldsymbol{R}(\boldsymbol{x}) \Rightarrow^{S} \boldsymbol{R}(\boldsymbol{x}+\mathbf{1}))] \hookleftarrow \\ \Rightarrow \boldsymbol{IC}^{S}[\forall^{S}\boldsymbol{x}(\boldsymbol{R}(\boldsymbol{x})]\}$$

It will be noted that the expression (8.4.21) is very similar to the expression (8.4.13) above, but without that expression's ambiguity; this demonstrates how logically coherent language can give rise to logically acceptable expressions.

(8.5) 'Higher Order Logic' and Assertion IV of Proposition V

Gödel's Proposition V is a proposition with two explicitly stated quantified variables, $\boldsymbol{R}$ and $\boldsymbol{X}$. We have already seen in Section (8.3) that the implication is that $\boldsymbol{R}$ is referenced by the expression $\exists \boldsymbol{PRF}^{\mathbf{R}}[\boldsymbol{PRF}^{\mathbf{R}}\ \boldsymbol{Proof}^{\mathbf{R}}\ (\boldsymbol{R}(\boldsymbol{X}))]$. There is no difficulty in expressing the proposition with a condition on the symbol $\boldsymbol{X}$, as in Section (8.4), rather than the implied assertion that the domain of $\boldsymbol{R}$ is the domain of all relations with the free variable $\boldsymbol{X}$. The assertion of Proposition V as given by Gödel in naïve format is:

(8.5.1) $\forall \boldsymbol{R}(\boldsymbol{x}), \forall \boldsymbol{x}, \exists \boldsymbol{u}, \exists \boldsymbol{r},$

(8.5.2) $\boldsymbol{R}(\boldsymbol{x}) \Rightarrow \boldsymbol{Bew}\{\boldsymbol{Sb}[\boldsymbol{r}, \boldsymbol{u}, \boldsymbol{Z}(\boldsymbol{x})]\}$

which can be expressed in a logically coherent manner to give:

(8.5.3) $\forall \boldsymbol{R}, \forall \boldsymbol{X}, \exists \boldsymbol{u}, \exists \boldsymbol{r}$

(8.5.4) $\boldsymbol{C}(\boldsymbol{R}, \boldsymbol{x}) \Rightarrow \exists \boldsymbol{PRF}^{\mathbf{R}}\{\boldsymbol{PRF}^{\mathbf{R}}\ \boldsymbol{Proof}^{\mathbf{R}}\ [\boldsymbol{Subst}\ (\boldsymbol{R}, \boldsymbol{x}, \boldsymbol{X})]\} \Rightarrow \boldsymbol{Bew}\{\boldsymbol{Sb}[\boldsymbol{r}, \boldsymbol{u}, \boldsymbol{Z}(\boldsymbol{X})]\}$

where $\boldsymbol{C}(\boldsymbol{R}, \boldsymbol{x})$ is the condition that $\boldsymbol{x}$ is a free variable in the relation $\boldsymbol{R}$, and there are no other free variables in $\boldsymbol{R}$. Note that in the above expression, the assertion that there exists a mapping function that maps the *'number-theoretic relations'* to the entities $\boldsymbol{r}$, $\boldsymbol{u}$ and $\boldsymbol{X}$ is not explicitly expressed.

We can also assert that the above applies for every relation $\boldsymbol{R}$ with one free variable, rather than only relations with the free variable $\boldsymbol{x}$, an assertion that is implied by Gödel's naïve expression of Proposition V:

(8.5.5) $\forall \boldsymbol{R}, \forall \boldsymbol{W}, \forall \boldsymbol{X}, \exists \boldsymbol{u}, \exists \boldsymbol{r},$

(8.5.6) $\boldsymbol{C}(\boldsymbol{R}, \boldsymbol{W}) \Rightarrow \exists \boldsymbol{PRF}^{\mathbf{R}}\{\boldsymbol{PRF}^{\mathbf{R}}\ \boldsymbol{Proof}^{\mathbf{R}}\ [\boldsymbol{Subst}\ (\boldsymbol{R}, \boldsymbol{W}, \boldsymbol{X})]\} \Rightarrow \boldsymbol{Bew}\{\boldsymbol{Sb}[\boldsymbol{r}, \boldsymbol{u}, \boldsymbol{Z}(\boldsymbol{X})]\}$

where $\boldsymbol{W}$ is a variable whose domain is symbols for variables that are defined for relations $\boldsymbol{R}$.

We now have a logically coherent derivation of an expression regarding formal system expressions, from an expression regarding *'number-theoretic relations'*, which is:

(8.5.7) $\forall \boldsymbol{R}, \forall \boldsymbol{X}, \exists \boldsymbol{u}, \exists \boldsymbol{r},$

$$\boldsymbol{C}(\boldsymbol{R}, \boldsymbol{x}) \Rightarrow \exists \boldsymbol{PRF}^{\mathbf{R}}\{\boldsymbol{PRF}^{\mathbf{R}}\ \boldsymbol{Proof}^{\mathbf{R}}\ [\boldsymbol{Subst}\ (\boldsymbol{R}, \boldsymbol{x}, \boldsymbol{X})]\}$$

(8.5.8) $$\Rightarrow \exists \boldsymbol{PRF}^{\mathbf{F}}\ [\boldsymbol{PRF}^{\mathbf{F}}\ \boldsymbol{Proof}^{\mathbf{F}}\ \boldsymbol{Subst}(\boldsymbol{FRM}^{\mathbf{F}}, \boldsymbol{v}^{\mathbf{F}}, \boldsymbol{X})]$$

and combining the above with those from Section (8.2) above, we have:

(8.5.9) $\forall \boldsymbol{R}, \forall \boldsymbol{X}, \exists \boldsymbol{u}, \exists \boldsymbol{r},$

(8.5.10) $\boldsymbol{C(R, x)} \Rightarrow \exists \boldsymbol{PRF}^{\mathbf{R}}\{\boldsymbol{PRF}^{\mathbf{R}}\ \boldsymbol{Proof}^{\mathbf{R}}\ [\boldsymbol{Subst}\ (\boldsymbol{R}, \boldsymbol{x}, \boldsymbol{X})]\}$

(8.5.11) $\Rightarrow \exists \boldsymbol{PRF}^{\mathbf{F}}\ [\boldsymbol{PRF}^{\mathbf{F}}\ \boldsymbol{Proof}^{\mathbf{F}}\ \boldsymbol{Subst}(\boldsymbol{FRM}^{\mathbf{F}}, \boldsymbol{v}^{\mathbf{F}}, \boldsymbol{X})]$

(8.5.12) $\Rightarrow \exists \boldsymbol{\Phi}(\boldsymbol{PRF}^{\mathbf{F}})\{\boldsymbol{\Phi}(\boldsymbol{PRF}^{\mathbf{F}})\ \boldsymbol{B}\ \boldsymbol{Sb}[\boldsymbol{\Phi}(\boldsymbol{FRM}^{\mathbf{F}}), \boldsymbol{\Psi}(\boldsymbol{v}^{\mathbf{F}}), \boldsymbol{\Phi}(\boldsymbol{X})]\}$

(8.5.13) $\Rightarrow \exists \boldsymbol{\Phi}(\boldsymbol{PRF}^{\mathbf{F}})\{\boldsymbol{\Phi}(\boldsymbol{PRF}^{\mathbf{F}})\ \boldsymbol{B}\ \boldsymbol{Sb}[\boldsymbol{\Phi}(\boldsymbol{FRM}^{\mathbf{F}}), \boldsymbol{\Psi}(\boldsymbol{v}^{\mathbf{F}}), \boldsymbol{Z}(\boldsymbol{X})]\}$

(8.5.14) $\Rightarrow \exists \boldsymbol{Y}\{\boldsymbol{Y}\ \boldsymbol{B}\ \boldsymbol{Sb}[\boldsymbol{T}, \boldsymbol{u}, \boldsymbol{Z}(\boldsymbol{X})]\}$

(8.5.15) $\Rightarrow \boldsymbol{Bew}\{\boldsymbol{Sb}[\boldsymbol{T}, \boldsymbol{u}, \boldsymbol{Z}(\boldsymbol{X})]\}$

where the expressions (8.5.9) - (8.5.15) are expressions of the language **PV**.

We note that there are now two possibilities. Either:

(8.5.16) Assertion III of Proposition V does not apply , so that it is not the case that for every recursive *'number-theoretic relation'* ***R*** with one free variable there exists a corresponding single formal expression with one free variable. This means that there does not exist one corresponding specific number ***T*** for every recursive *'number-theoretic relation'* ***R***

or

(8.5.17) Assertion III of Proposition V applies, so that for every recursive *'number-theoretic relation'* ***R*** with one free variable there is a corresponding single formal expression with one free variable, and there exists one corresponding specific number ***T*** for every recursive *'number-theoretic relation'* ***R***.

For any given formal system and definition of *'recursive number-theoretic relation'*, either (8.5.16) or (8.5.17) applies. Clearly, if (8.5.16) applies, then there is no basis for Proposition V. But in any case the argument up to this point is fundamentally flawed.

We now look at a simplified version of Gödel's theorem.

(9) A Simplified Version of Gödel's Theorem

A much simpler proof of Gödel's principal result that is a *'true but unprovable formula'*, (as given by his Proposition VI) can be obtained without any reference to recursion or ω–consistency, but which uses the principles of the outline of his proof, and hence is subject to the same fundamental flaw, which applies in the same way to Gödel's original proof and other versions of Gödel's proof (versions that use the same general methodology as Gödel's proof). We follow certain principles as used in Gödel's outline proof for his Proposition V, and for his Proposition VI. Again, we disregard here the fundamental flaw inherent in Gödel's proof that is the main subject of this paper.

(9.1.1) Given a number-theoretic relation ***R*(*x*)**, with the free variable ***x***, if there exists a definite method of transforming that relation to a formal system formula, then there exists a corresponding formal system formula ***F*(*v*)**, with the free variable ***v***.[9] The free variable ***v*** may be chosen arbitrarily.

(9.1.2) For this formal system formula ***F*(*v*)**, there exists a corresponding Gödel number ***r***, with a prime factor $\boldsymbol{p^{u}}$, where ***u*** is the number defined to correspond to the variable ***v***.

(9.1.3) So, given a Gödel number ***r***, with a prime factor $\boldsymbol{p^{u}}$, where ***u*** is the number defined to correspond to the variable ***v***, there exists a corresponding number-theoretic relation ***R*(*x*)**.[10]

(9.1.4) If the formal formula ***F*(*v*)** that corresponds to a number-theoretic relation ***R*(*x*)** is provable for a specific value of ***v***, it follows that the number-theoretic relation ***R*(*x*)** must be true for that value of ***x***, assuming consistency of the formal system.

(9.1.5) The expression *"there exists an expression A in the formal system that is a proof of the formal system formula B where its free variable has been substituted by some number"* has the corresponding number-theoretic relation ***Bew*{*Sb*[*r*, *u*, *Z*(*x*)]}**.

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[9] We note that this does not require that the number relation be recursive, only that there exists a definite method/algorithm of transforming the relation to a specific formula of whichever formal system is to be the subject of Gödel's proof. The relation we subsequently use is the relation **¬*Bew*{*Sb*[*x*, 17, *Z*(*x*)]}**, which is defined as **¬(∃*y* {*y B* [*Sb*[*x*, 17, *Z*(*x*)]})**. Gödel asserts that there exists a method of transforming the relation ***y B* [*Sb*[*x*, 17, *Z*(*x*)]** into a formula of the formal system, with a corresponding specific Gödel number. Since the relation **¬(∃*y* {*y B* [*Sb*[*x*, 17, *Z*(*x*)]})** only uses the additional symbols for a quantifier, a variable, negation, and parentheses, which are basic formal system symbols, it must also be also be possible to transform the relation into an expression in the formal system, with a corresponding specific Gödel number, by a method which disregards any meaning that may be given to the relation.

[10] If there is more than one such number-theoretic relation, one may always be selected, arbitrarily or for example, by alphabetical/alphanumeric ordering.

(9.1.6) For any formula of the formal system, either there exists a formal proof sequence of that formula or there does not. It follows that for any number $\boldsymbol{n}$ that is the Gödel number for a formal system formula, either $\boldsymbol{Bew(n)}$ or $\neg\boldsymbol{Bew(n)}$, and if the number $\boldsymbol{n}$ is not a Gödel number for a formal system formula, then $\neg\boldsymbol{Bew(n)}$. It follows that for any number $\boldsymbol{n}$, either $\boldsymbol{Bew(n)}$ or $\neg\boldsymbol{Bew(n)}$.

Since $\boldsymbol{Sb[w, u, Z(x)]\}}$ is a number-theoretic function which has a number value for any $\boldsymbol{w}$, $\boldsymbol{u}$ and $\boldsymbol{x}$, it follows that:
For all $\boldsymbol{w}$, for all $\boldsymbol{u}$, for all $\boldsymbol{x}$, either
(9.1.7) $\boldsymbol{Bew\{Sb[w, u, Z(x)]\}}$, or
(9.1.8) $\neg\boldsymbol{Bew\{Sb[w, u, Z(x)]\}}$

If $\boldsymbol{Bew\{Sb[w, u, Z(x)]\}}$ for some specific values of $\boldsymbol{w}$, $\boldsymbol{u}$, and $\boldsymbol{x}$, then any number-theoretic relation $\boldsymbol{R(x)}$ that corresponds to the number $\boldsymbol{w}$ holds for those values of $\boldsymbol{w}$, $\boldsymbol{u}$ and $\boldsymbol{x}$, so that we have:
For any specific values of $\boldsymbol{w}$, $\boldsymbol{u}$, and $\boldsymbol{x}$, either:
(9.1.9) $\boldsymbol{Bew\{Sb[r, u, Z(x)]\}} \Rightarrow \boldsymbol{R(x)}$, or
(9.1.10) $\neg\boldsymbol{Bew\{Sb[r, u, Z(x)]\}}$
where $\boldsymbol{R(x)}$ is a number-theoretic relation that corresponds to the number $\boldsymbol{w}$.

(9.1.11) We define a relation with one free variable $\boldsymbol{x}$ as $\neg\boldsymbol{Bew\{Sb[x, 17, Z(x)]\}}$, and we define $\boldsymbol{q}$ as a corresponding Gödel number for this relation $\neg\boldsymbol{Bew\{Sb[x, 17, Z(x)]\}}$ [11]

Substituting **17** for $\boldsymbol{u}$, and $\boldsymbol{q}$ for $\boldsymbol{w}$ gives:
For any specific value of $\boldsymbol{x}$, either:
(9.1.12) $\boldsymbol{Bew\{Sb[q, 17, Z(x)]\}} \Rightarrow \boldsymbol{R(x)}$, or
(9.1.13) $\neg\boldsymbol{Bew\{Sb[q, 17, Z(x)]\}}$
where $\boldsymbol{R(x)}$ is a number-theoretic relation that corresponds to the number $\boldsymbol{q}$.

Since $\neg\boldsymbol{Bew\{Sb[x, 17, Z(x)]\}}$ is a number-theoretic relation that corresponds to the number $\boldsymbol{q}$, it follows that:
For any specific value of $\boldsymbol{x}$, either:
(9.1.14) $\boldsymbol{Bew\{Sb[q, 17, Z(x)]\}} \Rightarrow \neg\boldsymbol{Bew\{Sb[x, 17, Z(x)]\}}$, or
(9.1.15) $\neg\boldsymbol{Bew\{Sb[q, 17, Z(x)]\}}$

[11] The number $\boldsymbol{q}$ is selected so that the corresponding formal formula contains as a free variable the formal variable that corresponds to the number **17**.

For $\boldsymbol{x = q}$ either:

(9.1.16) $\boldsymbol{Bew\{Sb[q, 17, Z(q)]\} \Rightarrow \neg Bew\{Sb[q, 17, Z(q)]\}}$, or

(9.1.17) $\boldsymbol{\neg Bew\{Sb[q, 17, Z(q)]\}}$

(9.1.18) It follows that (9.1.16) cannot apply, since that is a straightforward contradiction. Therefore it must be the case that (9.1.17) applies. This means that the relation $\boldsymbol{\neg Bew\{Sb[q, 17, Z(q)]\}}$ holds. Since that is the case, then there cannot be a formal proof of the formal formula that corresponds to $\boldsymbol{Sb[q, 17, Z(q)]}$.

Now the formal formula that corresponds to $\boldsymbol{Sb[q, 17, Z(q)]}$ is the formal formula that corresponds to the relation $\boldsymbol{R(q)}$. But the relation $\boldsymbol{R(q)}$ is actually the relation $\boldsymbol{\neg Bew\{Sb[q, 17, Z(q)]\}}$, which does hold true, as given by (9.1.17).

Therefore the relation $\boldsymbol{R(q)}$ holds true, and the formal formula that corresponds to $\boldsymbol{R(q)}$ must also be true, but there cannot be a formal proof of the formal formula that corresponds to $\boldsymbol{R(q)}$.

This completes the proof.[12] It will be observed that the above proof requires no additional assumptions that are not of the same nature as those used in Gödel's original proof.

In this simplified version, we note that the relation $\boldsymbol{R(q)}$ is asserted at the same time to be equivalent to both the expression $\boldsymbol{Sb[q, 17, Z(q)]}$ and the expression $\boldsymbol{\neg Bew\{Sb[q, 17, Z(q)]\}}$. But $\boldsymbol{Sb[q, 17, Z(q)]}$ is a function that represents a specific number value, whereas $\boldsymbol{\neg Bew\{Sb[q, 17, Z(q)]\}}$ is supposedly an expression that does not have a number value. Such anomalies are also present in Gödel's original version, but are not so readily apparent as they are here.

[12] Note that, unlike Gödel's Proposition V, we cannot assert that the converse of (9.1.16), that is, we cannot assert that $\boldsymbol{\neg Bew\{Sb[q, 17, Z(q)]\} \Rightarrow Bew\{Sb[q, 17, Z(q)]\}}$.

(10) The Inherent Flaw in Gödel's Theorem

From the above simplified version of Gödel's proof it is clear that the flaw in Gödel's proof does not lie in the details concerning recursion and **ω**–consistency, since the result of a *'true but unprovable formula'* can be produced without such details. We note that the principle of the following argument refers equally to the simplified version of Gödel's proof given here, and to Gödel's original proof.

(10.1) Language, Variables and Specific Values

The fundamental flaw in Gödel's proof is essentially very simple, and revolves around the failure to clearly distinguish the meta-language and its sub-languages. This is shown by looking at the elementary concepts of propositions and variables referred to by the propositions of the proof. In Gödel's paper, as is the norm, the concept of *'variable'* is taken for granted, without any need envisaged for clarification or definition of what is meant by *'variable'*. But the existence of variables implies that there are two types of entity referenced by relational operators: variables and non-variables. We note that the domain of any variable cannot be that variable itself – the domain of a variable is a domain of values that are not variables, in that particular language. For any proposition with a quantifier on a variable, either the universal quantifier, *"For all…"*, or the existential quantifier, *"There exists some…"*, that proposition can only imply a proposition where the variable is substituted by a specific value of the domain of that variable. For example, the proposition $\forall x(x > 3)$ might imply **4 > 3**, **5 > 3**, etc [13] but it cannot imply $x > 3$ since $x > 3$ is not a proposition since it contains a free variable. Nor can it imply, for example, $y > 3$, where y is also a variable of the language being used.

In the mapping of an expression from some sub-language **Ln** to another sub-language **Lm**, we expect that the definition for a mapping will define the correspondence of the variables of the language **Ln** to the variables of the language **Lm**, which for any given mapping is a one-to-one correspondence. Since there are infinitely many such variables for any given language **Ln**, the definition of a general correspondence that is applicable to all relations must be by an expression that includes a variable that has the domain of all variables of the language **Ln**, and a variable that has the domain of all variables of the language **Lm**.

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[13] Depending on the rules and axioms of whatever system is being applied.

Thus in general, we have some mapping $\boldsymbol{T}^{\mathbf{Ln\text{-}Lm}}$, so that $\boldsymbol{X}^{\mathbf{Ln}} = \boldsymbol{T}^{\mathbf{Ln\text{-}Lm}}(\boldsymbol{Y}^{\mathbf{Lm}})$ is an expression where $\boldsymbol{X}^{\mathbf{Ln}}$ is a variable of the meta-language with the domain of variables of the language **Ln**, and $\boldsymbol{Y}^{\mathbf{Lm}}$ is a variable of the meta-language with the domain of variables of the language **Lm**. Clearly, in such an expression, the variables of the language **Ln** and of the language **Lm** are not variable quantities of the meta-language, but are specific values of the meta-language. That is, for some particular variable of the language **Ln**, that we call $\boldsymbol{v}^{\mathbf{Ln}}$, there is some particular variable of the formal language **Lm** that we call $\boldsymbol{v}^{\mathbf{Lm}}$, and we can assert that $\boldsymbol{v}^{\mathbf{Ln}} = \boldsymbol{T}^{\mathbf{Ln\text{-}Lm}}(\boldsymbol{v}^{\mathbf{Lm}})$. Since $\boldsymbol{v}^{\mathbf{Lm}}$ is not a variable in the meta-language, and since $\boldsymbol{v}^{\mathbf{Ln}} = \boldsymbol{T}^{\mathbf{Ln\text{-}Lm}}(\boldsymbol{v}^{\mathbf{Lm}})$, it follows that $\boldsymbol{v}^{\mathbf{Ln}}$ also is not a variable in the meta-language.

Gödel's Proposition V is a proposition that relies on other propositions for its proof. None of these propositions can have any free variables. A clear distinction between variables of these propositions and the specific values that they represent must apply to all those propositions. And since all those propositions all have to be in the same logical language, there must be a clear distinction between the variables of that language and the specific values of that language.

In Gödel's Proposition V, where he says, *'For every recursive number-theoretic relation for which there exists n free variables, there exists a corresponding number r, and for every x,* **…***'*, that is a proposition with bound variables. One variable is *'recursive number-theoretic relation'*, which is bound by *'For every … '* and another is *'number'*, which is bound by *'there exists … .'*

When Gödel refers to *'n free variables'*, the word *'variables'* in that proposition is itself a variable. It is a variable in the language of the proposition because it is not a specific value. It represents a variable of a *'number-theoretic relation'*. And it is a bound variable, because it is bound by *'there exists …'* Since the word *'variables'* is actually a variable in the language of Gödel's Proposition V, then there can also be propositions in that language which refer to some specific value that may be represented by that variable, that is a variable of a *'number-theoretic relation'*. Any such values are not themselves variables in that language.

(10.2) Meta-Language and *'Number-Theoretic Relations'*

Up to this point, we have avoided as far as possible directly observing any contradictions arising from the reasoning indicated by Gödel's outline proof of Proposition V. Section (8.5) above gave a logical derivation of an expression of the language **PV** concerning entities of the formal language from an expression of the language **PV** concerning entities of a *'number-theoretic relation'*. By ignoring the correct formulation and declining to furnish a detailed proof of Proposition V, Gödel conveniently avoided the issue of how the mapping from the relation ***R*** to the formal system formula $\boldsymbol{FRM}^{\mathbf{F}}$ is to be defined. The outline proof simply implies that, given a *'number-theoretic relation'*, a corresponding formal system expression may be derived from that relation, without considering at all the manner in which that might be achieved.

The relation is merely a combination of symbols, and the formal system expression is simply a combination of symbols. If the formal system expression can in principle be derived from the relation, then it must be possible in principle to define every step in the mapping of the initial symbol combination to the final symbol combination. The mapping of a relation to a formal formula is therefore a process that in principle is capable of full definition, provided that the concept of *'number-theoretic relation'* can in principle be fully defined. It is irrelevant that there may in principle be many different such definable mappings.

As referred to above, Proposition V begins with *'For every recursive number-theoretic relation …'*¸ which means that *'number-theoretic relation'* is a variable in that language, which means that the specific values to which that variable refers to are specific *'number-theoretic relations'*. So that means that in the language of Gödel's Proposition V, *'number-theoretic relations'* are specific values. Similarly, the reference to *'n free variables'* means that variables of *'number-theoretic relations'* are specific values in that language.

This means that the language of Gödel's Proposition V is a language in which *'number-theoretic relations'* are specific values, and in which the variables of *'number-theoretic relations'* are specific values. It is also the case that in the language of Gödel's Proposition V formal formulas are also specific values, and the variables of formal formulas are specific values. That means that the language of Gödel's Proposition V is a meta-language to both the formal language and to *'number-theoretic relations'*.

The fundamental flaw in Gödel's theorem is that it confuses the meta-language and the language of *'number-theoretic relations'*, so that although the language of *'number-theoretic relations'* must be a sub-language, Gödel refers to *'number-theoretic relations'* with the assumption that such expressions are expressions of the meta-language.

(10.3) Symbols for Relational Operators

Up to this point we have largely ignored the implications of using the same symbol for relational operators in a meta-language and a sub-language. Since a meta-language can be chosen to have different symbols for relational operators from the sub-languages that it refers to, if Gödel's result depends on the same symbol being used for a relational operator in the meta-language and a sub-language, then clearly the result is a result that is dependent on confusion between the meta-language and the sub-language.

In the following, we assume that a meta-language is chosen that does not have the same relational operators as its sub-languages. However, it should be noted that the following argument does not rely on this aspect. The following argument still applies to the following if we allow the same relational operators for the meta-language and sub-languages, ***except*** that any references to a formal formula is to be taken as referring to a formal formula that contains a formal system variable (free or bound), and any references to a *'number-theoretic relation'* is to be taken as referring to a *'number-theoretic relation'* expression that contains a *'number-theoretic relation'* variable (free or bound).

Since the formal language is a sub-language, formal formulas are simply values in the meta-language, being simply combinations of symbols that the meta-language refers to. The same applies to *'number-theoretic relations'*, which similarly are simply values in the meta-language, being simply combinations of symbols that the meta-language refers to. Simply because the meta-language might be able to find a corresponding formal formula for a given *'number-theoretic relation'*, that cannot indicate that either the formal expression or the *'number-theoretic'* expression are expressions in the syntax of the meta-language other than as specific values to be referenced by relational operators.

In order for the meta-language of Gödel's Proposition to have a function that can give a formal formula for a given *'number-theoretic relation'*, it must use variables to represent the specific values which are the symbols of formal formulas and *'number-theoretic relations'*. Any value that is a specific value of any of those variables cannot be a variable of that meta-language, and that means that the variables of formal formulas and *'number-theoretic relations'* cannot be variables of that meta-language.

While it is obvious that the formal system is a sub-language to the language **PV**, it is now also clear that the language of any relation defined by ***R*** is also a sub-language to the language **PV**. Once it is appreciated that the language of Gödel's' Proposition V is a meta-language to the language of *'number-theoretic relations'*, it is evident that the proposition is irredeemably flawed.

(10.4) Examples of Contradictions arising from Gödel's Proof

(10.4.1) **Contradictions arising from assertions of truth rather than provability**
Gödel's proof refers to *'number-theoretic relations'* being true or false, while formal formulas are referred to as being provable or not provable. While Gödel never actually defines what he means by a *'number-theoretic relation'*, one would expect that if formulas of the formal language can be expressions regarding numbers, that such expressions would be *'number-theoretic relations'*. But even if by some definition formal formulas could be excluded from the definition of *'number-theoretic relations'*, Gödel gives no reason as to why one should refer to expressions of one sub-language (formal formulas) as being provable or not provable, and those of another sub-language (*'number-theoretic relations'*) as being true or false. (See also Appendix 2: Provability and Truth, and Appendix 3: The Formal System and 'Number-Theoretic Relations')

In the language of Gödel's Proposition V formal formulas and *'number-theoretic relations'* are simply combinations of symbols, which are specific values, so it is absurd to refer to these specific values as being true or false. For example, Gödel refers to expressions such as **¬*Bew*{*Sb*[*x*, 17, *Z*(*y*)]}**, asserting that it is a *'number-theoretic relation'* which has a matching Gödel number. He also asserts that it must be either true or false. But that is a contradiction. If it is a *'number-theoretic relation'* then it is seen by the meta-language as a specific value. It is not an expression that has a syntactical relationship to any other expression of the meta-language other than as a specific value; it is simply a combination of symbols to which the meta-language cannot assign a determination of true nor false other than by a determination of the existence of a set of *'number-theoretic relations'* for which an appellation of 'proof' might be applied, which would allow a term 'provable' to be applied to the combination of symbols in question.

On the other hand, if the expression is not a *'number-theoretic relation'* then Proposition V cannot give a corresponding Gödel number, for the assertion is that for every *'number-theoretic relation'* there is a matching Gödel number.

(10.4.2) **Contradictions arising from assertions regarding *'number-theoretic relations'***
When Gödel refers to the expression **¬*Bew*{*Sb*[*X*, 17, *Z*(*X*)]}** in Proposition V, he refers to the symbol ***X*** in this expression as a variable of the meta-language, bound by *'for every **X**, **…**'*. But if ***X*** is a variable of the meta-language, then it cannot be a variable of a *'number-theoretic relation'*. And if ***X*** cannot be a variable of a *'number-theoretic relation'*, then the expression **¬*Bew*{*Sb*[*X*, 17, *Z*(*X*)]}** cannot be a *'number-theoretic relation'*. But that is a contradiction, because **¬*Bew*{*Sb*[*X*, 17, *Z*(*X*)]}** has to be a *'number-theoretic relation'* so that there can be a matching Gödel number.

But if the expression ***Bew*{*Sb*[*X*, 17, *Z*(*X*)]}** is an expression of the language **PV**, then all of its variables, free or bound, will be variables of the language **PV**. It also means that it cannot be an expression that is a *'number-theoretic relation'*, since expressions that are *'number-theoretic relations'* are expressions of a sub-language of **PV**, and whose symbols for variables are specific values in the language **PV**, and not variables in the language **PV**.

That means that when a relation ***R*** is defined as **¬*Bew*{*Sb*[*X*, 17, *Z*(*X*)]}** as above, and it is asserted that it is a *'number-theoretic relation'*, that means that its variables (free or bound) cannot be variables of the language **PV**, since the language of *'number-theoretic relations'* is a sub-language to the language **PV**. That means that when it is defined that such an expression may be such a relation ***R***, that is a contradiction. And that contradiction serves to confirm the flaw in the logical argument of Gödel's theorem.

The confusion in Gödel's theorem is exacerbated by the use of the same terminology ***Bew***, ***Sb***, and ***Z*** being used for expressions of the language **PV** and the sub-language of *'number-theoretic relations'*. But it is not only the free variables which must be different in such expressions for the language **PV** and the sub-language. Consider an expression such as **¬*Bew*{*Sb*[536, 17, *Z*(536)]}**. In the sub-language of *'number-theoretic relations'*, the bound variables would be variables of the language of *'number-theoretic relations'*. But in the meta-language **PV** the same variables of the same expression would, according to Gödel, be variables of that language **PV** and which are not variables of the language of *'number-theoretic relations'*.

As noted previously, for the expression of Proposition V, there is no reason in principle why the language **PV** should not have different symbols for relational operators and well as different symbols for variables, to those of the language of *'number-theoretic relations'* that is referenced by ***R*** in the proposition, for the reasons given in Section (8.4). In that case, for an expression that has no variables at all, it is not ambiguous whether that expression is an expression of the language **PV** or of a sub-language, unlike the case in Gödel's naïve expression of Proposition V.

We note that Gödel's proof language **PV** itself implies that expressions in the formal system cannot be considered to be *'number-theoretic relations'*, while similar expressions that consist of only of the same relational operators as the formal system and number symbols (that is, ¬, ∀, **)**, **(**, ∨, and ***f*** and **0**) and symbols for variables that are different to those defined for the formal system can be considered to be *'number-theoretic relations'*. (See also Appendix 3: The Formal System and 'Number-Theoretic Relations')

(10.4.3) **Contradictions arising from misunderstanding of 'Higher Order' logic:**
In the naïve expression of Gödel's proof, when an expression such as $\neg$***Bew*{*Sb*[*x*, *n*, *Z*(*x*)]}** is defined and it is asserted that it is a *'number-theoretic relation'* and which is a relation referred to by ***R*(*x*)** in Proposition V, that is on the basis that:
"For all ***R*(*x*)**, ***Expression*[*R*(*x*)]**" implies the expression obtained when the free variable ***R*(*x*)** of ***Expression*** has been substituted by a permissible specific value for that variable.

That means that when Gödel asserts that the value $\neg$***Bew*{*Sb*[*x*, *n*, *Z*(*x*)]}** is a value that may be referred to by ***R*(*x*)** in his Proposition V, that means that it may be substituted for any instance of the variable ***R*(*x*)** in ***Expression***. It also means that in that new ***Expression***, that value of ***R*(*x*)** cannot change, since it is a specific value. However, according to Gödel, ***x*** is still a variable in the expression $\neg$***Bew*{*Sb*[*x*, *n*, *Z*(*x*)]}**.

But if that is the case, and it is also the case that $\neg$***Bew*{*Sb*[*x*, *n*, *Z*(*x*)]}** is a specific value, that implies that it is immaterial what value is subsequently given for ***x***. So if it is the case that when ***x* = *q*** it can be said that $\neg$***Bew*{*Sb*[*q*, 17, *Z*(*q*)]}** applies (for example, as in (9.1.18) above), then that would imply that $\neg$***Bew*{*Sb*[*x*, 17, *Z*(*x*)]}** for every value of ***x***. And that is an absurdity.

(10.4.4) **Contradictions arising from equivalence of the Φ Function and the *Z* Function**
As in (8.2.11) above, Proposition V relies on a claim of equivalence of the Gödel numbering function and the ***Z*** function (Gödel's relation 17). That is, it must be asserted that for any value of ***X*** that is a number, **Φ(*X*) ≡ *Z*(*X*)**. Now, that expression cannot be a *'number-theoretic'* expression, since the Gödel numbering function refers to formal language symbols that are not numbers. So the assertion that **Φ(*X*) ≡ *Z*(*X*)** must be an assertion of the meta-language of Proposition V.

But besides the variable ***X***, the definition of **Φ(*X*)** includes other variables as bound variables, variable that may represent values other than numbers. That means that even if the value of ***X*** is a number, the Gödel numbering function is still not a *'number-theoretic relation'*. For example, **Φ(5234)**, **Φ(905781)** and **Φ(135874)** are not *'number-theoretic relations'*. But when Gödel claims that **Φ(5234)** is equivalent to ***Z*(5234)**, and if ***Z*(5234)** can be referred to by the meta-language as a *'number-theoretic relation'*, then **Φ(5234)** must also be a *'number-theoretic relation'*. Otherwise they cannot be equivalent in the meta-language of Proposition V. But **Φ(5234)** cannot be a *'number-theoretic relation'*. That is a contradiction, caused by the failure to clearly delineate the meta-language, the formal language, and *'number-theoretic relations'*.

Alternatively, we may consider the claim of equivalence of the Gödel numbering function and the $\boldsymbol{Z}$ function is as follows:

The expression $\boldsymbol{Bew\{Sb[T, 17, Z(X)]\}}$ is asserted to be a *'number-theoretic relation'*, where $\boldsymbol{X}$ and $\boldsymbol{T}$ are variables with the domain of natural numbers. At the same time, it is asserted as in (8.2.14) - (8.2.17).that:

(10.4.5) $\boldsymbol{Bew\{Sb[\Phi(FRM^F), \Psi(v^F), \Phi(X)]\} \Rightarrow Bew\{Sb[T, 17, Z(X)]\}}$

where it is asserted that $\boldsymbol{T}$ is the number given by the Gödel mapping function $\boldsymbol{\Phi}$ on some formal system formula $\boldsymbol{FRM^F}$, that is that $\boldsymbol{T = \Phi(FRM^F)}$, and that **17** is the number given by the Gödel mapping function $\boldsymbol{\Psi}$, that is that $\boldsymbol{17 = \Psi(v^F)}$.

But if it is asserted that:

(10.4.6) $\boldsymbol{Bew\{Sb[\Phi(FRM^F), \Psi(v^F), \Phi(X)]\} \Rightarrow Bew\{Sb[T, 17, Z(X)]\}}$, and also that

(10.4.7) $\boldsymbol{Bew\{Sb[T, 17, Z(X)]\} \Rightarrow Bew\{Sb[\Phi(FRM^F), \Psi(v^F), \Phi(X)]\}}$

then equivalence is being asserted, i.e., it is being asserted that

(10.4.8) $\boldsymbol{Bew\{Sb[T, 17, Z(X)]\} \equiv Bew\{Sb[\Phi(FRM^F), \Psi(v^F), \Phi(X)]\}}$

And if equivalence of the two expressions is being asserted then the expression $\boldsymbol{Bew\{Sb[\Phi(FRM^F), \Psi(v^F), \Phi(X)]\}}$ must also satisfy the definition of a *'number-theoretic relation'*, even though it includes variables that do not have the domain only of natural numbers (or, if the variables are substituted, includes values that are not numbers).

Now $\boldsymbol{Bew\{Sb[\Phi(FRM^F), \Psi(v^F), \Phi(X)]\} \Rightarrow Bew\{Sb[T, 17, Z(X)]\}}$ must be asserted as in (8.2.14) - (8.2.17)[14]. That means that to avoid a contradiction in the definition of a *'number-theoretic relation'*, it would have to be asserted that:

$\boldsymbol{Bew\{Sb[T, 17, Z(X)]\}}$ does not imply $\boldsymbol{Bew\{Sb[\Phi(FRM^F), \Psi(v^F), \Phi(X)]\}}$.

But that of course, would mean that we cannot assert that $\boldsymbol{T = \Phi(FRM^F)}$ and $\boldsymbol{17 = \Psi(v^F)}$; that is, that we cannot assert that the Gödel numbering system is a mapping function (or the $\boldsymbol{\Psi}$ function) at the same time as defining $\boldsymbol{Bew\{Sb[T, 17, Z(X)]\}}$ as a *'number-theoretic relation'*.

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14 Again, due to the vagueness of the outline that Gödel gives for Proposition V, this is not explicitly stated in Gödel's proof, but it is necessarily implied by the notion of a mapping from the formal system by means of the Gödel numbering function.

(10.4.9) **Contradictions arising from failure to define *'number-theoretic relations'***
As for the above, the same applies to any number that is a Gödel number or a number given by the function **Ψ**. Since for any such number ***x*** in the meta-language, either
***x* = Ψ(Ω)**, or ***x* = Φ(Ω)** where **Ω** is some symbol or combination of symbols of the formal system.
That means that in the meta-language, any expression that includes such a number ***x*** is equivalent to the expression obtained when that number ***x*** is substituted by **Ψ(Ω)** or **Φ(Ω)** as appropriate. And that means that the notion that those expressions can be termed *'number-theoretic relations'* is absurd since they are equivalent to expressions that refer to entities other than numbers.

Similarly, if a *'number-theoretic relation'* ***R*** is defined as **¬*Bew*{*Sb*[*q*, 17, *Z*(*q*)]}**, and if for some formal formula ***FRM***, ***q* = Φ[*FRM*]**, and if there is a function **λ** that gives a formal formula ***FRM*** that corresponds to a 'number theoretic relation' ***R***, then we have ***FRM* = λ(*R*)**, so that we have:
***R* = ¬*Bew*{*Sb*[*q*, 17, *Z*(*q*)]}**
and
***FRM* = λ(*R*) = λ(¬*Bew*{*Sb*[*q*, 17, *Z*(*q*)]})**
and
***q* = Φ[*FRM*] = Φ[λ(¬*Bew*{*Sb*[*q*, 17, *Z*(*q*)]})]**
so that
***R* = ¬*Bew*{*Sb*[Φ[λ(¬*Bew*{*Sb*[*q*, 17, *Z*(*q*)]})] , 17, *Z*(Φ[λ(¬*Bew*{*Sb*[*q*, 17, *Z*(*q*)]})])]}**
and
***R* = ¬*Bew*{*Sb*[Φ[λ(*R*)] , 17, *Z*(Φ[λ(*R*)]}**

where ***R*** is defined in terms of itself, and where ***R*** is supposedly a *'number-theoretic relation'*. But since ***R*** is equivalent to the above expression, that expression must also be a *'number-theoretic relation'*, which of course is absurd. It is an absurdity caused by the inherent self-reference generated by Gödel's proof, a 'proof' that asserts the existence of some function **λ** at the same time as declining to define it.

(11) Summary

Gödel's proof fails because the expressions referred to in Gödel's proof as *'number-theoretic relations'* cannot be referred to as *'number-theoretic relations'* by the meta-language of his Proposition V. As soon as it is asserted that a symbol used as a variable in one of these *'number-theoretic relations'* is a variable of his proof language, that expression cannot be defined as a *'number-theoretic relation'*, however that might be defined by that proof language.

The failure to realise the confusion of meta-language and sub-language that is the fundamental flaw of Gödel's proof arises from a myriad of interdependent deficiencies:

- the failure to define precisely a *'number-theoretic relation'* as referenced by Gödel's Proposition V.
- the failure to give a detailed proof for Proposition V, which is the key proposition on which Proposition VI depends.
- the reference to an expression of a sub-language as 'true' when it should be a reference to its provability.
- the use of a meta-language that uses the same symbols for relational operators as its sub-languages.

The net result of these failings is a failure to clearly delineate the meta-language of Proposition V and its sub-languages, which is the fundamental flaw of Gödel's theorem. This flaw results in an expression that is asserted to be an expression of the meta-language and a sub-language at the same time, which is a logical contradiction.

The intuitive 'proof' of Proposition V, a proof that defines 'number-theoretic relations' as including the expressions concerning numbers used in the proof itself, is a 'proof' that is made by the intuitive reference to an expression as an integral syntactical part of the language at the same time as referring to it as an expression of a sub-language. The fundamental incompatibility of these two modes of reference within a single proof language is overlooked, an oversight that is due the reliance on intuition. The failure to recognise this is at the root of the misunderstanding of Gödel's proof. The reason why the flaw in Gödel's proof has been overlooked for so long must in part be attributed to the confusion generated by vagueness of presentation of Gödel's proof, and which gives the lie to the commonly accepted consensus that Gödel's proof is a masterpiece of logical argument. On the contrary, it is a masterpiece of obfuscation.

Gödel's proof superficially appears to state that that proof itself could not be expressed in a formal language, and in so doing, appears to absolve itself from the rigour required of formal languages. But how could Gödel's proof be verified if that was the case? For any proof, any verification that the propositions of that proof are to be considered proven can only be made in a supervisory language. And for a supervisory language to verify a proof it must verify each step of the proof, and in order to do that, it must consider the rules that apply to the language of the proof. When those rules are not explicitly stated that does not mean that no examination of those rules can be made, for there must be rules that are implied by various assumptions.

The erroneous assumption in Gödel's proof is that because a supervisory language can attest to the validity of an expression in one proof language, and can also attest to the validity of an expression in a different proof language, that an expression that simply combines two such expressions must be a valid expression in some proof language. This intuitive notion is incorrect. If it is asserted that a combined expression is a valid expression in some proof language, then the supervisory language must examine that expression and the language of that expression, and consider if it is a valid expression according to the rules of the proof language of that expression. A supervisory language must reject the notion that the self-reference generated by Gödel's Proposition VI is a valid result, relying as it does on the flawed Proposition V.

No-one to date has given a satisfactory explanation as to why there cannot be a logically coherent formalisation of Gödel's argument. Once the fundamental flaw in Gödel's argument is known, it is obvious why this must be the case – there cannot be such a logically coherent formalisation, since any attempt at such a formalisation would clearly demonstrate the inherent contradiction. Unsurprisingly, this results from the failure to acknowledge the distinction between variables of the proof language and variables of the languages referenced by the proof language.

Of course, there will be formal formulas that refer to numbers which are their own Gödel number by the Gödel numbering function. But that doesn't mean that the formal formula is talking about itself. The Gödel numbering function cannot be expressed by the formal system itself, since the numbering system has to refer to the variables of the formal language as specific values.

Gödel's assertion that the ***Z*** function, which gives the correct Gödel number for any given number, is in some sense equivalent to the Gödel numbering function, for values that are only numbers, is also a concept that cannot be expressed in the formal language.

The notion that a formal formula, when it refers to some number, can discern that that number is its own Gödel number is erroneous. The formal language is completely oblivious of any numbering system applied to it. And this applies to any language that can be completely defined so that a numbering system can be applied to it .

(12) Conclusion

Many observers have maintained that Gödel's proof was a paradox, whereas many others insisted that it was not. Since "paradox" is simply a euphemism for "contradiction", Gödel's result is a paradox and is a contradiction. There are unacceptable underlying assumptions that are reliant on intuition in the language of Gödel's paper that give rise to that result. By demonstrating that those assumptions are untenable, the apparent conundrums arising from Gödel's paper disappear.

The belief that Gödel's proof represents some sort of mathematical 'truth' is completely and utterly wrong. It is apparent that Gödel's result is, after all, just the same as all other proofs, that is, it is a result that depends on the assumptions and rules used to generate it. Conventionally, there are proofs that we accept as 'correct', and proofs that we do not accept as 'correct'. According to accepted standards, Gödel's proof cannot be considered to express some sort of indispensable universal fundamental 'truth', and the assumptions that generate Gödel's result are completely unacceptable by any commonly accepted standards of logic.

Far from being a singular and monumental landmark in logic that will remain visible far in space and time[c], Gödel's proof is a travesty of the principles of logical deduction, an argument where at the crucial point, logic is cast aside in favour of intuition. The intense irony is that Gödel's result has been heralded as demonstrating that formal reasoning is inferior to intuitive reasoning[d], when the entire basis of Gödel's result is itself due to flawed intuition, and actually demonstrates the pitfalls that can result from the uncritical use of intuition. The notion that Gödel's proof demonstrates some sort of universal truth that differentiates 'truth' and 'provability' has led to some wonderfully bizarre statements, such as *"Gödel showed that provability is a weaker notion than truth"*,[e] and *"It follows that no machine can be a complete or adequate model of the mind, that minds are essentially different from machines… We can never, not even in principle, have a mechanical model of the mind"*.[f]

It transpires that Wittgenstein, who was ridiculed for refusing to accept the result of Gödel's theorem, had valid grounds for so doing, even if he could not establish a specific rebuttal of Gödel's theorem. In fact, what is commonly referred to as Wittgenstein's 'notorious paragraph'[g] about Gödel's proof, where he says, *"…we must also ask, 'true' in what system?"* shows that Wittgenstein was correct. Gödel's *'true but not provable'* formula is only *'true'* in a proof language that makes certain crucial assumptions which cannot be considered to be acceptable assumptions for a valid proof language. It is interesting to note that the assumptions inherent in Gödel's proof which lead to contextual ambiguities being ignored are referred to in Wittgenstein's Tractatus[h] where he notes, *"In this way [where the same word has different modes of signification] the most fundamental confusions are easily* *produced - the whole of philosophy is full of them."* Wittgenstein has been quoted as having referred to Gödel's theorems as 'logical tricks', and judged by the standards of conventional mathematics and logic, this is indeed what they are – the superficial complexities of the first part of Gödel's proof have served to hide the underlying intuitive assumptions of Gödel's Proposition V, which once revealed are seen to be trivially absurd. It is rather ironic that Gödel has always been considered to be the logician par excellence, while Wittgenstein's writings have frequently been dismissed as being vague and impenetrable, when it is the vagueness and impenetrability of Gödel's Incompleteness Theorem that has obscured its inherent flaw for so long. And although Wittgenstein was belittled for not 'understanding' Gödel's Theorem, it is notable that he cautioned that, *"In order to avoid such errors we must make use of a sign-language … by not using the same sign for different symbols and by not using in a superficially similar way signs that have different modes of signification"*. [h]

Gödel's proof has been accepted because superficially it appears to give the desired result. Contrary to popular opinion, Gödel's result was not a revolutionary overthrowing of the commonly accepted philosophy that existed at the time of its publication – it was accepted by the vast majority of philosophers and logicians precisely because it appeared to say what they wanted to hear – in effect, that man is capable of a higher level of reasoning than a formal system, that there are realities that cannot be captured by a formal system, and that human intuition is superior to formal reasoning. Since those that opposed it at the time were unable to demonstrate any flaw in the proof, their misgivings about the theorem were ignored.

Many logicians have stressed the importance of the fact that Gödel's proof generates a specific formal expression, but in principle this demonstrates nothing. It ignores the blatantly obvious fact that **every** erroneous proof generates a fallacious expression. Gödel's specific formula is simply a result that can be generated in a language that allows ambiguity of reference and is simply a reminder of the old adage, "Rubbish in, rubbish out". Contrary to commonly held notions, there is no inherent self-reference in the formal system. It is not at all surprising that Gödel's result arises not from the formal system, but from the ambiguities inherent in Gödel's proof language. The notion that somehow there could be such a self-reference that could be valid and at the same time avoid an infinitely circular self-referencing loop was always bizarre, and remains so. It could have not been put better than by Gödel himself in his paper, where he noted the analogy between his result, Richard's antinomy, and the liar paradox. Those paradoxes are all based on an infinitely circular self-reference, so that they can never be clearly defined. The irony is that Gödel did not consider it be illogical to suppose that his self-referencing statement was neither an infinitely circular self-reference nor the result of ambiguity in its derivation.

It is a sobering thought that 75 years have passed since Gödel's proof was published. In that time, it has been the subject of intense study. The failure to uncover the naïve assumptions that give rise to that result is surely the most risible failure in the entire history of logic and mathematics. The failure to perceive what Wittgenstein could perceive, that Gödel's result had to indicate some sort of 'logical trick', is surely the most worrying failure in the entire history of logic and mathematics. The failure to desist from directing the most unwarranted levels of derision and contempt at anyone who dared to question Gödel's proof is surely the most embarrassing failure in the entire history of logic and mathematics.

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[a] Turing, Alan: "On Computable Numbers, with an Application to the Entscheidungsproblem", Proceedings of the London Mathematical Society, ser.2.vol. 42 (1936-7), pp.230-265 (corrections, Ibid, vol. 43 (1937) pp. 544-546)

[b] Gödel, Kurt: "On Formally Undecidable Propositions of Principia Mathematica and Related Systems", 1931, translation by B. Meltzer, Oliver & Boyd, Edinburgh, 1962

[c] Von Neumann, John: Speech, 1951, *"Kurt Gödel's achievement in modern logic is singular and monumental - indeed it is more than a monument, it is a landmark which will remain visible far in space and time"*

[d] Von Neumann, John: Letter to Carnap, June 1931, *"There is no more reason to reject intuitionism…"*

[e] Hofstader, Douglas: "Gödel, Escher, Bach, An Eternal Golden Braid", Penguin Books, 1980

[f] Lucas, J.R: "Minds, machines, and Gödel", Philosophy, vol. 36 (1961), pp. 112–137

[g] Wittgenstein, Ludwig: "Remarks on the Foundations of Mathematics", ed. G.E. von Wright, R. Rees and G.E.M. Anscombe, trans. G.E.M. Anscombe (Cambridge: MIT, 1956): I, Appendix III, §8

[h] Wittgenstein, Ludwig: Section 3.3.2 of "Tractatus Logico-Philosophicus" translation by C. K. Odgen, Routledge, 1922

Appendix 1: Gödel's Prefatory Argument

The outline of the argument presented in Gödel's preface is all too frequently cited as though it were the proof proper, even though it can easily be shown to be untenable. Essentially, it proceeds as follows:

> "We designate a formula of a specified formal system with one free variable of natural numbers by the term *class-sign*. We assume that the *class-signs* can be arranged in a series, to give a one-to-one correspondence between the *class-signs* of the specified formal system and natural numbers. We denote the *class-sign* corresponding to the number ***n*** by ***F*(*n*)**, and we note that the concept of *class-sign* as well as the ordering function ***F*** are definable in the specified formal system." [15]

But if ***F*** is definable in the specified formal system, then there will be a combination of symbols that define the function ***F*** and which includes one or more occurrences of a symbol that is a free variable, which we call ***x***. When the single free variable ***x*** is substituted by a specific number ***n***, then according to the above, the resultant expression will be an expression of the formal system with at least one free variable, which is absurd. It follows that the function ***F*** cannot be defined within the specified formal system itself, and that the subsequent outline argument is unfounded.

Of course, the argument presented in the preface appears not to be the basis for the subsequent argument, since the one-to-one correspondence of formal system expressions to numbers given by the Gödel numbering system is defined outside of the formal system. Gödel's Proposition V appears at first glance to overcome this difficulty, but it only hides the impossibility of generating an expression with a free variable by the substitution of the free variable in an expression, a concealment dependent on intuition and which is entirely due to language confusion.

[15] In his preface, Gödel actually refers to Russell's formal system, Principia Mathematica as the specified formal system, but this makes no difference to the principle involved.

Appendix 2: Provability and Truth

When Gödel states in footnote 39 that *"Proposition V naturally is based on the fact that for any recursive relation **R**, it is decidable … from the axioms of the system P, whether the relation **R** holds or not."*, it appears at first glance that Gödel is relying on the relation ***R*** being *'decidable from the axioms of the system P'* rather than on the notion of truth.

But while Gödel avoids the use of the term *'true'* in relation to this proposition, the term is nonetheless implied by the Proposition and the following text. In the proposition, it is implied that if ***R*(*x*)** *'holds'* or is *'true'*, then ***Bew*{*Sb*[*r*, u_n, *Z*(*x*)]}** *'holds'* or is *'true'*, But there is no clear definition of what is being asserted by the unqualified reference to ***R*(*x*)**, or what is being asserted by the implied terms *'holds'*, and *'true'*.

Either the unqualified reference to the term ***R*(*x*)** is equivalent to stating that the relation ***R*(*x*)** is *'provable from the axioms of the system P'* or it is not. Or we might say, either the terms *'holds'* and *'true'* are equivalent to the term *'provable from the axioms of the system P'* or they are not.

According to what we are given by Gödel, it is clear that any *'number-theoretic relation'* which is *'provable from the axioms of the system P'* can be said to *'hold'*, or to be *'true'*. This means that if the terms *'hold'*, *'true'* and *'provable from the axioms of the system P'* are not equivalent, then it must be the case that there are *'number-theoretic relations'* which *'hold'* and are *'true'*, but are not *'provable from the axioms of the system P'* (and which are not themselves axioms).

But this means that, regardless of whatever definition is actually applied to the terms *'holds'* or *'true'*, it must be the case that there are *'number-theoretic relations'* which *'hold'* and are *'true'*, but are not *'provable from the axioms of the system P'*, and which are not axioms.

But if the initial assumption is that there are *'number-theoretic relations'* which *'hold'* and are *'true'*, but are not *'provable from the axioms of the system P'*, and which are not axioms, the rest of Gödel's proof is completely pointless. For if a proof asserts there exists some formula of the formal system, which is not an axiom, which is not *'provable from the axioms of the system P'*, but nonetheless *'holds'* and is *'true'*, and that assertion itself relies on the assumption that there exists at least one *'number-theoretic relation'*, which is not an axiom, which *'holds'* and is *'true'*, but is not *'provable from the axioms of the system P'*, then it proves nothing at all.

Appendix 3: The Formal System and 'Number-Theoretic Relations'

Whatever the definition of a *'number-theoretic relation'* might be, it must be the case that variables of the formal system cannot be included as variables of *'number-theoretic relations'* in that definition, without contradictions arising such as in the following:

Although Proposition V is expressed in terms of a variable $\boldsymbol{x}$, there is an implied assertion that the proposition applies for any valid variable of the language in which Proposition V is expressed. Hence if, for example, a variable of the formal system was defined as a valid variables of the language of *'number-theoretic relations'* defined by the entity $\boldsymbol{R}$, we would have, given Gödel's expression of Proposition V:

(A.1.1) $\forall \boldsymbol{R(v_1)}, \forall \boldsymbol{v_1}, \exists \boldsymbol{FRM}^{\mathrm{F}}, \exists \boldsymbol{r},$

(A.1.2) $\boldsymbol{R(v_1)} \quad \Rightarrow \exists \boldsymbol{PRF}^{\mathrm{F}}\ [\boldsymbol{PRF}^{\mathrm{F}}\ \boldsymbol{Proof}^{\mathrm{F}}\ \boldsymbol{Subst}(\boldsymbol{FRM}^{\mathrm{F}}, \boldsymbol{v}^{\mathrm{F}}, \boldsymbol{v_1})]$

(A.1.3) $\quad \Rightarrow \boldsymbol{Bew}(\ \boldsymbol{Sb}[\boldsymbol{r}, \boldsymbol{u}, \boldsymbol{Z}(\boldsymbol{v_1})])$

where $\boldsymbol{v_1}$ is a variable of the formal language, a variable of *'number-theoretic relations'* and a variable of the language **PV** of the proposition.

Proposition V asserts that the variable $\boldsymbol{v}^{\mathrm{F}}$ (and the corresponding $\boldsymbol{u}$) may be chosen arbitrarily so that $\boldsymbol{v_1}$ could be chosen as the variable $\boldsymbol{v}^{\mathrm{F}}$. But if the variable $\boldsymbol{v_1}$ is a bound variable in the proposition, if $\boldsymbol{v}^{\mathrm{F}}$ is chosen to be $\boldsymbol{v_1}$, the following expression is a *'proposition'*:

(A.1.4) $\forall \boldsymbol{R(v_1)}, \forall \boldsymbol{v_1}, \exists \boldsymbol{FRM}^{\mathrm{F}}, \exists \boldsymbol{r},$

(A.1.5) $\boldsymbol{R(v_1)} \Rightarrow \exists \boldsymbol{PRF}^{\mathrm{F}}\ [\boldsymbol{PRF}^{\mathrm{F}}\ \boldsymbol{Proof}^{\mathrm{F}}\ \boldsymbol{Subst}(\boldsymbol{FRM}^{\mathrm{F}}, \boldsymbol{v_1}, \boldsymbol{v_1})]$

(A.1.6) $\quad \Rightarrow \boldsymbol{Bew}(\ \boldsymbol{Sb}[\boldsymbol{r}, \boldsymbol{u_1}, \boldsymbol{Z}(\boldsymbol{v_1})])$

where the *'variable'* $\boldsymbol{v_1}$ is substituted by the *'variable'* $\boldsymbol{v_1}$.

The above basically demonstrates what the naïve expression of Proposition V is really asserting in its *'intuitive'* format, which is:

(A.1.7) $\forall \boldsymbol{R(x)}, \forall \boldsymbol{x}, \exists \boldsymbol{r},$

(A.1.8) $\exists \boldsymbol{PRF}^{\mathrm{R}}\ [\boldsymbol{PRF}^{\mathrm{R}}\ \boldsymbol{Proof}^{\mathrm{R}}\ \boldsymbol{Subst}(\boldsymbol{R}, \boldsymbol{x}, \boldsymbol{x})]$

(A.1.9) $\quad \Rightarrow \exists\ \boldsymbol{y}\{\ \boldsymbol{y}\ \boldsymbol{B}\ \boldsymbol{Sb}[\boldsymbol{r}, \boldsymbol{u}, \boldsymbol{Z}(\boldsymbol{x})]\}$

where the *'variable'* $\boldsymbol{x}$ is substituted by the *'variable'* $\boldsymbol{x}$. The confusion is generated by the naïve expression of Gödel's Proposition V, and by Gödel referring to entities referenced by the entity $\boldsymbol{R}$ as *'number-theoretic relations'*, although any definition of which must exclude all expressions of the formal system, although the formal expressions that are expressions concerning numbers constitute what we would normally regard as the most fundamental *'number-theoretic relations'*.