THE IMPOSSIBILITY OF REPRESENTATION OF A GÖDEL NUMBERING FUNCTION BY A FORMULA OF THE FORMAL SYSTEM

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v1 16 Sept 2013

Abstract

In many incompleteness proofs there is a claim that, given a Gödel numbering function that encodes sequences of symbols of a given formal system, there is a formula of that formal system that corresponds to that Gödel numbering function such that the formula itself can refer unambiguously to formulas of that formal system. This paper proves that this cannot be the case.

1 Introduction

It can be observed that many proofs of incompleteness rely on the proposition that, given a Gödel numbering function that encodes sequences of symbols of a certain formal system, there must be a formula of that formal system that corresponds to that Gödel numbering functions such that the formula can refer unambiguously to formulas of that formal system. Often the validity of the proposition is simply assumed, in other cases, there is an erroneous proof of the proposition.

There are several examples of such proofs that rely on such a proposition, see for example [1], [2], [3], [4].[a] Some incompleteness proofs rely on the ’diagonal lemma’, which also relies on such a proposition, for example see [9] or [10].

This paper proves that such a proposition is invalid and that a formula of the formal system as asserted by such a proposition is an impossibility.

[a]For an analysis of the errors therein, see [5],[6],[7],[8].

2 Overview of Gödel numbering functions

The standard description of a Gödel numbering function is as follows:

First we have a function ψ that gives a one-to-one correspondence between some symbol and some natural number. So we might have, for example:

Formal Symbol	**Corresponding number**
0	$\psi[0] \equiv 2$
S	$\psi[S] \equiv 3$
$)$	$\psi[)] \equiv 5$
$($	$\psi[(] \equiv 7$
$\neg$	$\psi[\neg] \equiv 9$
$\vee$	$\psi[\vee] \equiv 11$
$\forall$	$\psi[\forall] \equiv 13$
$=$	$\psi[=] \equiv 15$

For a given sequence of symbols, this gives, by application of the ψ function, a series of number values. The second step is to apply another function to this series. This function takes each of these number values in sequence; for the n^{th} such value, the n^{th} prime number is raised to the power of that value (the value given by the ψ function), and this gives another series of number values. The final step is to take all of these values and multiply them together. This now gives a single number value. Given any sequence of the given set of symbols, there is a corresponding Gödel numbering for that sequence, a number that is unique for that sequence; the Gödel numbering preserves the uniqueness of the sequences, each sequence having one corresponding number, for example:

Formal Expression	**Corresponding Gödel number**
0	2^2
$SSS0$	$2^3 \cdot 3^3 \cdot 5^3 \cdot 7^3 \cdot 11^2$
$\neg(SSSS0 = SS0)$	$2^9 \cdot 3^7 \cdot 5^3 \cdot 7^3 \cdot 11^3 \cdot 13^3 \cdot 17^2 \cdot 19^{15} \cdot 23^3 \cdot 29^3 \cdot 31^2 \cdot 37^5$

We note here a point that is often overlooked in considerations of these matters, and that is that the definition of a Gödel numbering function is independent of any formal system. A Gödel numbering function is simply a function whose free variable has a domain of values that are sequences of symbols. The fact that some of these sequences may be formulas of a formal system is irrelevant to the definition of a Gödel numbering function.

3 The impossibility of representation of a Gödel numbering function in the object language

Theorem. *It is not possible for a formula of a formal system to include the definition of a Gödel numbering system such that the formula can unambiguously refer to the formulas of that formal system.*

Proof. Given a certain set of symbols, for example 0, S,), (, ¬, ∨, ∀, there can be many different Gödel numbering functions that apply to the same set of sequences, since different exponents can be assigned arbitrarily to the symbols. These different Gödel numbering functions will have different values for any particular sequence.

Let GN_A be one such Gödel numbering function, which assigns the exponents 3, 5, 7, 9, 11, 13, 15 respectively to the symbols 0, S,), (, ¬, ∨, ∀. Given the sequence

$0S00 \vee S \neg \forall 0$

the Gödel numbering function GN_A will have the value:

$2^3 \cdot 3^5 \cdot 5^3 \cdot 7^3 \cdot 11^{13} . 13^5 \cdot 17^{11} \cdot 19^{15} \cdot 23^3$

Let GN_B be another Gödel numbering function which assigns the exponents 5, 3, 7, 9, 11, 13, 15 respectively to the symbols 0, S,), (, ¬, ∨, ∀. For the same sequence of symbols, the Gödel numbering function GN_B will have the value:

$2^5 \cdot 3^3 \cdot 5^5 \cdot 7^5 \cdot 11^{13} \cdot 13^3 \cdot 17^{11} \cdot 19^{15} \cdot 23^5$

while for the sequence of symbols

$S0SS \vee 0 \neg \forall S$

the Gödel numbering function GN_B will have the value:

$2^3 \cdot 3^5 \cdot 5^3 \cdot 7^3 \cdot 11^{13} \cdot 13^5 \cdot 17^{11} \cdot 19^{15} \cdot 23^3$

which we note is the same value that the function GN_A gives for the different sequence $0S00 \vee S \neg \forall 0$ (as in the above), that is, the value of $GN_A[0S00 \vee S \neg \forall 0]$ is precisely the same value as $GN_B[S0SS \vee 0 \neg \forall S]$.

We now assume that, given a certain formal system, there are two distinct formulas $gn_a(x)$ and $gn_b(y)$ of that formal system that correspond to the functions GN_A and GN_B, and which include the information of the definition of the Gödel numbering functions GN_A and GN_B, and which can unambiguously refer to the formulas of that formal system.[b]

It follows that if $gn_a(x)$ and $gn_b(y)$ correspond to the Gödel numbering functions $GN_A[X]$ and $GN_B[Y]$, there must be some correspondence of each value of x to each sequence X, and of each value of y to each sequence Y. Disregarding considerations as to how such a correspondence might be defined,[c] we now consider the claim that the purported formal system formulas $gn_a(x)$

[b] For convenience the formulas $gn_a(x)$ and $gn_b(y)$ are assumed to be functions, but the same principles apply for relations where a variable of the relation corresponds to the free variable of the Gödel numbering function and another variable corresponds to the value of the Gödel numbering function.

[c] It might be noted that the nature of the correspondence of the purported formulas $gn_a(x)$ and $gn_b(y)$ to the Gödel numbering functions is conspicuous by its absence in proofs that claim there is such a formula, as is the actual representation of such a formula.

or $gn_b(y)$ can refer unambiguously to the formulas of that formal system itself. Since $gn_a(x)$ and $gn_b(y)$ are formulas of a formal system, the actual format by which they are represented is immaterial; that is, the value of the function is the same for any given value of the free variable, regardless of whatever format the formula is represented in.

Given a particular set of symbols, there are several distinct fundamental representations of any formal system that uses those symbols. Given a specific formal system, there can be one representation which we refer to as (i), and which includes the symbols 0, S,), (, ¬, ∨, ∀ (as above). There can be another representation, which we refer to as (ii), and which is identical except that the positions of the symbols 0 and S are reversed. That is, wherever a 0 occurs in representation (i), a S appears in that position in representation (ii), and wherever a S occurs in representation (i), a 0 appears in that position in representation (ii).

This means that for any sequence X that is a formula of the formal system in representation (i), there is a sequence in representation (ii), which we denote as X', that represents the same formula.

Clearly the function $GN_A[X]$ will not have the same value as $GN_A[X']$.

But the assumption is that $gn_a(x)$ is a representation of the function $GN_A[X]$, so that if X is a sequence that is a formula of the formal system, then x is a specific value that corresponds to X, and the formula $gn_a(x)$ refers in some way to the formula given by the sequence X.

However, X is not of itself a formula, it is simply a sequence of symbols. It is only within the context of a certain formal system that the sequence X has the properties of a formula of that system. Since the formula $gn_a(x)$ corresponds to the function $GN_A[X]$, the corresponding x only corresponds to a sequence of symbols, and not to any particular formula. This applies regardless of whether the formula $gn_a(x)$ is in representation (i) or representation (ii) or any other representation. Similarly, since the value of the function $GN_A[X]$ is independent of whether the sequence X is a formula of some formal system, the numerical value of the purported formula $gn_a(x)$ must also be independent of whether the sequence X is a formula of some formal system.

This gives rise to a contradiction. On the one hand, the numerical value of the formula $gn_a(x)$, for any given value of x, must be independent of the actual symbols used for its own representation. On the other hand, since the assumption is that it can refer unambiguously to a formula of its own formal system, the numerical value of the formula $gn_a(x)$ for any given value of x must be dependent on whatever actual representation is chosen for the formal system (including that formula $gn_a(x)$ itself) so that for any given value of an x that corresponds to some X, when the formula $gn_a(x)$ is represented in representation (i) the formula $gn_a(x)$ has one specific value, but when it is represented in representation (ii) it has a different value for precisely the same value of its variable x.

This contradiction means that the assumption that a formula of the formal system can contain the information of the definition of the Gödel numbering function such that it can refer unambiguously to the formulas of that formal system is unsustainable.

References

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