On the Reality of the Continuum and Russell's Moment of Candour

James R Meyer

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Abstract

An article entitled 'On the Reality of the Continuum' by Anne Newstead and James Franklin, which was published in *Philosophy* in 2008, contains a serious error of logic. This paper analyses that error and indicates how a logical resolution can be achieved.

In a paper '*The continuum: Russell's moment of candour*', Christopher Ormell questions whether a comment by Bertrand Russell[6] reveals an admission by Russell that his belief in the existence of 'indefinable' real numbers is not entirely based on logical considerations, and questions some of the arguments regarding that issue.[A] In an article '*On the Reality of the Continuum*'[3] Anne Newstead and James Franklin voice their disagreement with Ormell, where they state:

> *'No one disputes that when the real numbers are restricted to definable numbers, there are only countably many.*[B] *The core philosophical issue concerns whether all numbers must be definable. In what follows, it will be useful to bear in mind the dialectic between constructivists and classical mathematicians.*
>
> *Classical mathematicians happily accept the inference:*
>
> (1) *There are uncountably many real numbers.*
>
> (2) *There are at most countably many definable numbers.*
>
> (3) *Therefore, not all numbers are definable.*[C]
>
> *Constructivists reject step* (3) *of the argument, because they hold that properly understood, all real numbers are definable. The general inference* (*from* (1) *and* (2) *to* (3)) *is valid. Furthermore, no one disputes* (2).'

[A] Ormell has also questioned arguments regarding 'indefinable' real numbers elsewhere, see '*Can we understand uncountability*'[5] and '*Some Varieties of Superparadox*'.[4]

[B] Newstead and Franklin include a footnote: *Indeed, since definable numbers must be defined using countably many symbols, the countability of definable numbers follows immediately.*

[C] Newstead and Franklin include a footnote: *One could further infer from (3) using classical logic that (4) There exist indefinable real numbers. Some constructivists would also reject the inference to step (4) from (3),because the transition* $\neg\forall x Fx \rightarrow \exists x \neg Fx$ *is not valid in intuitionist logic. However, even constructivists who maintain classical logic will want to reject the argument on the basis of step (3).*

The claims that:

> *'No one disputes that when the real numbers are restricted to definable numbers, there are only countably many.'*
>
> and
>
> *'...no one disputes [that there are at most countably many definable numbers]'*

are claims that there must exist a function that enumerates all definable numbers. But according to the conventional presentation of Cantor's diagonal proof[2] this gives rise to a contradiction, since the diagonal proof states that, given a set of real numbers and a function that enumerates the members of that set, then another number (the 'diagonal' number) can **always** be defined in terms of that enumeration function, where the n^{th} digit of that number is given by taking the n^{th} digit of the n^{th} number in the enumeration and replacing that digit by a specified different digit.[D]

But this is contradictory, since if there is an enumeration function of all definable numbers, then that diagonal number, since it is definable, must already be given as the m^{th} real number in the enumeration for some natural number m. But at the same time, the definition of the diagonal number specifies that the m^{th} digit of that number must be different to the m^{th} digit of that number itself. This is impossible. This contradiction can also be obtained by noting that the diagonal number must be different to every number enumerated by the enumeration function, yet the assumption is that the enumeration function enumerates all definable numbers, hence it must include the diagonal number, which again is impossible.

The only way to avoid this contradiction is to acknowledge that there are two distinct types of enumeration functions of real numbers:

(a) Enumeration functions from which a diagonal number can be defined, and

(b) Enumeration functions from which a diagonal number cannot be defined.

An example of (a) is the function $f(n) = \sqrt{n} - \lfloor \sqrt{n} \rfloor$ that gives the non-integer part of the square root of n. The diagonal number can be defined in terms of this function, and is different to every number in the enumeration.

That leaves case (b), an enumeration function from which a diagonal number cannot be defined. We note that combinations of symbols, of themselves, do not define anything. It is only within the context of a specific language system with a clearly defined alphabet that a combination of symbols can be said to define some specific concept. A given combination of symbols can represent one definition in one language system, but represent an entirely different definition in another language system.

We now suppose that there is a set of symbol combinations defined in terms of a specific language system A and where at least some of those combinations represent real numbers in that language system A. We might suppose that the set is enumerable, since the symbol combinations can be enumerated by a dictionary style method. But to avoid the contradiction noted above, then it must not be possible to define a diagonal number from that enumeration. This can indeed be the case if the enumeration is defined in a language system B which is a different language system to the language system A, and where the language system B cannot evaluate those symbol combinations as real numbers. In other words, the symbol combinations given by the enumeration have no meaning

[D] This also applies to other proofs of non-denumerability such as Cantor's 1874 proof[1] and the power set proof, which is a version of the diagonal proof.

as real numbers in the language system B (the language system of the enumeration function). Hence there can be an enumeration of all definable real numbers where it is also the case that a diagonal number cannot be defined from that enumeration.

Given such an enumeration, we can review the three assertions considered previously: [E]

(1) There is no function that can enumerate all real numbers.

(2) There is a function that can enumerate all definable real numbers.

(3) Therefore, not all numbers are definable.

Any assumption that, even though a real number cannot be defined from an enumeration, a diagonal number nevertheless 'exists' in a Platonist sense would be to assume from the outset that there exist real numbers that cannot be defined. Such an assumption must be rejected, as it assumes that which one is attempting to prove.

Statement (1) is a statement that is based on the initial assumption that if there were an enumeration function of all real numbers, then there must exist a diagonal number in terms of that that enumeration function, which results in a contradiction which has led people to assume that the initial assumption was incorrect. This paper has shown that that contradiction indicates only that at least one of the prior assumptions was incorrect, and has shown that a different assumption was the cause of the contradiction.

If it is the case that there can be an enumeration function that can enumerate all definable real numbers of a given mathematical system, but where no diagonal number can be defined from that enumeration, there is no rational reason to assume that such an enumeration function cannot be an enumeration of all real numbers. Given the acknowledgment that there must necessarily be two types of enumeration function in order to avoid any contradiction, there is no logical rationale whatsoever for the assumption that there cannot be a mathematical system that can express all real numbers, and there is no logical rationale for statement (1).

From that it follows that there is also no logical basis for statement (3), and the entire structure of Newstead and Franklin's argument lacks any logical implication.

References

[1] Georg Cantor. Ueber eine Eigenschaft des Inbegriffs aller reellen algebraischen Zahlen. *Journal für die reine und angewandte Mathematik*, 77:258–262, 1874.

[2] Georg Cantor. *Über eine elementare Frage der Mannigfaltigkeitslehre*. Druck und Verlag von Georg Reimer, 1892.

[3] Anne Newstead and James Franklin. On the Reality of the Continuum Discussion Note: A Reply to Ormell,'Russell's Moment of Candour'. *Philosophy*, 83(1):117–127, 2008.

[E] Here the three assertions are stated without including the term 'countably many' since the use of this term at the outset is incorrect. The notion of different 'levels' of 'many' such as 'countably many' and 'uncountably many' are notions that themselves depend on the assumption that the three original statements, here rephrased, are logically tenable.

[4] Christopher Ormell. *Some Varieties of Superparadox*, volume 2 of *Studies in the Meaning of Mathematics*. Mathematics Applicable Group, School of Education, University of East Anglia, Norwich in association with Ashby Anthologies, 1993.

[5] Christopher Ormell. Can we understand uncountability? *The Mathematical Gazette*, 92(524):252–256, 2008.

[6] Bertrand Russell. *My philosophical development*. Psychology Press, 1995.