A FUNDAMENTAL FLAW IN INCOMPLETENESS PROOFS BY S. C. KLEENE

James R Meyer
http://www.jamesrmeyer.com

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Abstract

This paper examines proofs of incompleteness and related matters by S.C. Kleene and demonstrates that Kleene's conclusions rely on unproven assumptions and errors of logic. It is also shown that alternative methods of proof of Kleene's propositions are similarly erroneous.

1 Introduction

In this paper we examine two papers by Kleene 'General recursive functions of natural numbers' and 'Recursive Predicates And Quantifiers' and show that Kleene's conclusions rely on unproven assumptions and errors of logic. Note that in the following we will use $GN(\omega)$ to represent the Gödel numbering function, so that $GN(\omega)$ indicates the Gödel number of the formal formula ω.

2 Kleene's paper 'General recursive functions of natural numbers'

In this paper[1] Kleene defines a series of recursive number-theoretic functions, including the following:

> Supposing the function $\psi(n,x,y)$ given, we define a series of functions as follows:

$$\begin{aligned}
\psi(0,x,y) &= x \\
\psi(n+1,x,y) &= \psi(n,x,y) \\
\lambda(0,z) &= l(z) \\
\lambda(k+1,z) &= [k+l]\cdot\lambda(k,z)^2 \\
\tau(0,z) &= z \\
\tau(k+1,z) &= \ldots
\end{aligned}$$

Kleene then states:

> Then if z or $\tau(0,z)$ is the Gödel number for the sequence S_o of the $\lambda(0,z)$ numbers $z_1,\ldots,z_l,(z_1,\ldots,z_l>0)$, $\tau(k+1,z)$ is the Gödel number for the sequence S_{k+1} of the $\lambda(k+1,z)$ numbers $\psi(n,x,y)$, for $n=0,\ldots,k$ and x and y and ranging over S_k, in a certain order.

Here Kleene makes an error which is lamentably common in incompleteness proofs. He confuses the meta-language and the language of number-theoretic expressions. In the above he asserts that '$\tau(k+1,z)$ is the Gödel number for the sequence ...'. The use of natural language here serves to obfuscate what Kleene is actually attempting to assert. In strict mathematical language it will be observed that Kleene is attempting to assert:

$$\tau(k+1,z)=GN(\omega) \tag{2.1}$$

where ω is some expression of the formal system.

Clearly, the above equation 2.1 cannot be a number-theoretic expression, since the right hand side of the equation is not a number-theoretic expression.

Furthermore, a language that is a meta-language to number-theoretic expressions will have variables that refer to the symbols of the language of number-theoretic expressions, and hence will have the domain of symbols of that language of number-theoretic expressions. Clearly then, the meta-language cannot have a variable which is the same symbols as any symbol of the language of the number-theoretic expressions it refers to - if the meta-language is a logically valid meta-language. And this means that the equation 2.1 cannot be a logically valid expression of a language that is a meta-language to number-theoretic expressions.

The logical conclusion is that the equation 2.1 is not a valid expression of either a number-theoretic expression, or of a meta-language, and that Kleene's conclusions, which rely on this illogical expression, have no logical mathematical validity.

3 Kleene's paper 'Recursive Predicates And Quantifiers'

In this paper[2], Kleene's crucial assertion is his Enumeration Theorem (Theorem I in his paper). Essentially this asserts that there is a number-theoretic predicate (T_n in Kleene's paper) with $n+1$ free variables that enumerates number-theoretic predicates R with n free variables. The enumeration is by the additional free variable which can take any natural number value. The assertion of the theorem is given by Kleene as:

> *Given a general recursive predicate $R(x_1,\ldots,x_n,y)$, there are numbers f and g such that:*[a]

$$\exists y, R(x_1,\ldots,x_n,y) \equiv \exists y, T_n(f,x_1,\ldots,x_n,y) \tag{6}$$

$$\forall y, \neg R(x_1,\ldots,x_n,y) \equiv \forall y, \neg T_n(g,x_1,\ldots,x_n,y) \tag{7}$$

[a] Note that Kleene omits the implied assertion that there exists such a primitive recursive predicate T_n for every such predicate R.

Kleene's argument for the above theorem is as follows:

> We introduce a metamathematical predicate ξ_n (for each particular n) as follows: $\xi_n(Z, x_1, \ldots, x_n, Y) : Z$ *is a system of equations, and* Y *is a formal deduction from* Z *by* $R1$ *and* $R2$ *of an equation of the form* $\mathtt{f}(\mathtt{x_1}, \ldots, \mathtt{x_n}) = \mathtt{x}$, *where* $\mathtt{f}$ *is the principal function symbol of* Z, *where* $\mathtt{x_1}, \ldots, \mathtt{x_n}$ *are the numerals representing the natural numbers* $x_1, \ldots, x_n$ *and where* $\mathtt{x}$ *is a numeral.*

Kleene asserts that this gives:

$$\exists y, R(x_1, \ldots, x_n, y) \equiv \exists y, \xi_n(F, x_1, \ldots, x_n, Y) \tag{4}$$

$$\text{and} \quad \forall y, \neg R(x_1, \ldots, x_n, y) \equiv \forall Y, \neg \xi_n(F, x_1, \ldots, x_n, Y) \tag{5}$$

Kleene goes on to assert:

> Using Gödel's idea of arithmetizing meta-mathematics, suppose that natural numbers have been correlated to the formal objects, distinct numbers to distinct objects. The metamathematical predicate $\xi_n(Z, x_1, \ldots, x_n, Y)$ is carried by the correlation into a number-theoretic predicate $S_n(z, x_1, \ldots, x_n, y)$, the definition of which we complete by taking it as false for values of z, y not both correlated to formal objects. For a suitably chosen Gödel numbering, we can show, with a little trouble that S_n is primitive recursive. Now (4) translates under the arithmetization into:
>
> $$\exists y, R(x_1, \ldots, x_n, y) \equiv \exists y, S_n(f, x_1, \ldots, x_n, y) \tag{6a}$$
>
> with f as the Gödel number of the system of equations F. The formula
>
> $$\forall y, \neg R(x_1, \ldots, x_n, y) \equiv \forall y, \neg S_n(g, x_1, \ldots, x_n, y) \tag{7a}$$
>
> is obtained likewise from (5), after changing the notation so that R is interchanged with $\neg R$. We shall go over from S_n to a new predicate T_n. The predicate T_n is defined from S_n as follows.
>
> $$T_n(z, x_1, \ldots, x_n, , y) \; : \; S_n(z, x_1, \ldots, x_n, y) \;\&\; \forall t, [t < y \implies \neg Sn(z, x_1, \ldots, x_n, t)]$$

Kleene continues, asserting that the primitive recursiveness of T_n follows from that of S_n, and asserts that the formulas (6) and (7) of the theorem follow from (6a) and (7a) by the definition of T_n in terms of S_n.

In the above, Kleene asserts that meta-mathematical predicate, ξ_n is *'carried by correlation'* into a number-theoretic predicate. But he doesn't provide any logical argument to support this assertion. He fails to provide any proof that a purely number-theoretic predicate can refer to a Gödel numbering function that references number-theoretic predicates. And in any proof, such as Kleene's, where the goal is to produce a statement that refers to statements of its own language, the crucial step in the proof is to prove that the formal system in question can actually reference itself. But Kleene, rather than providing a rigorous proof of his assertions, relies on a vague claim that a meta-mathematical statement is *'carried by correlation'* to a statement of the formal system.

Moreover, since the meta-mathematical predicate ξ_n must use at least one variable to refer to all the symbols of whatever language of number-theoretic relations is used, there is at least one variable in the meta-mathematical predicate ξ_n for which there cannot be a corresponding variable in ***any*** expression of the given language of number-theoretic relations.[b]

The consequences of Kleene's failure to provide a rigorous proof of his assertions can be seen in Kleene's perfunctory attempt at justification, which contains obvious errors of logic. When he refers to the 'arithmetization' of his equation (4) to the equation (6a), he refers to his equation 6a by:

$$\exists y, R(x_1,\ldots,x_n,y) \equiv \exists y, S_n(f,x_1,\ldots,x_n,y)$$
with f as the Gödel number of the system of equations F.

Here, Kleene's use of natural language 'f as the Gödel number of ... ', serves to obfuscate the inherent illogicality of his assertions. If we attempt to formulate his assertions in strict logical form, since Kleene is asserting the equivalence of f and $GN(\omega)$, we would get:

$$\exists y, R(x_1,\ldots,x_n,y) \equiv \exists y, S_n(GN(\omega),x_1,\ldots,x_n,y) \tag{3.1}$$

where ω is an expression of the formal system.

But Kleene has already asserted that S_n is a number-theoretic predicate. As such the variables of S_n must have the domain of natural numbers or number-theoretic functions. Clearly, the Gödel numbering function GN is not a natural number nor is it a number-theoretic function. This means that the equation 3.1 is not a logically valid expression. Kleene's error here is regrettably common in attempts at proof of incompleteness, where it is assumed that, because the Gödel numbering function evaluates as a natural number value, it can be a valid member of the domain of a variable of a number-theoretic expression. The equivalence of a single property of two entities, in this case the numerical value of two expressions, does not imply that all properties possessed by the two entities are identical - such as being a member of the domain of a certain variable. The same error also occurs in later sections of Kleene's paper.

[b]The same omission also occurs further on in Kleene's paper; in a proof of his 'Normal form theorem', Kleene again introduces a meta-mathematical predicate, $\mathfrak{U}$. He then states:

> *'By the Gödel numbering already considered, the meta-mathematical function* $\mathfrak{U}(Y)$ *is carried into a number-theoretic function* $U(y)$*.'*

Again, Kleene completely sidesteps the need for a proof of this conjecture that a meta-mathematical function can be 'carried into' a number-theoretic function. As in the Enumeration theorem, Kleene blithely ignores the need for rigor in his paper, and his conclusions are not proven with the standard of rigor required of a mathematical proof.

4 Alternative proofs of Kleene's theoerms

There are claims that Kleene's theorems can be proved using the notion of 'Unlimited Register Machines' (URMs). An 'Unlimited Register Machine' is defined as a machine that can run URM programs and which has, among other things, unlimited memory (see, for example [3]).

A feature of these 'proofs', is not surprisingly, a version of Gödel numbering. Clearly, if the programs for a URM are generated from a fixed set of symbols, then all the programs can be listed in an alphabetical style. And so the URM programs can each be assigned a unique number, in the same way that formal formulas can be assigned Gödel numbers. However, it should be noted that such a numbering function is a function that is in a language that is a meta-language to the language of URM programs, in the same way that a Gödel numbering function is in a language that is a meta-language to the formal language that the function refers to.

This enumeration of URM programs is employed in proofs of Kleene's assertions that use the notion of URMs. In such 'proofs', there will be an assertion such as:

'Let $\boldsymbol{e} = \gamma(P)$ *be the code number of the URM program* $\boldsymbol{P}$*.'*

The proof then asserts that a URM program takes, as its input, an expression containing the symbol $\boldsymbol{e}$ as referred to by $\boldsymbol{e} = \gamma(P)$. But the URM ***cannot*** refer to the function γ, since that function is, by definition, in a language that is a meta-language to the language of URM programs - so it is not expressible in the whatever language is used for these URM programs. And that means that this URM program language cannot access the information in the γ function. This is precisely the same situation as we have already encountered - a formal system cannot access the information within a Gödel numbering function for that formal system.

5 Conclusion

In common with many other authors of incompleteness 'proofs', Kleene relies on unproven assumptions and illogical statements to arrive at his conclusions. The superficial facade of mathematical content cannot conceal Kleene's failure to provide a mathematically rigorous argument to support his assertions. Such a lack of mathematical rigor is abhorrent, and it renders Kleene's conclusions mathematically unacceptable.

References

[1] S. C. Kleene. General recursive functions of natural numbers. *Mathematische Annalen*, 112:727–742, 1936.

[2] S. C. Kleene. Recursive predicates and quantifiers. *Transactions of the American Mathematical Society*, 53:41–73, 1943.

[3] John C. Shepherdson and H. E. Sturgis. Computability of recursive functions. *Journal of the Association of Computing Machinery (JACM)*, 10:217–255, 1963.