The Fundamental Flaw in
Gödel's Proof of the Incompleteness Theorem
"On Formally Undecidable Propositions of
Principia Mathematica and Related Systems"

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1 Abstract

This paper identifies the fundamental error inherent in Gödel's proof of his Incompleteness Theorem. The error is generated by the ambiguity of the language of Gödel's outline proof of his Proposition V, a proposition for which Gödel declined to furnish a detailed proof. The error arises from a confusion of the meta-language and the languages to which it refers, a confusion which is exacerbated by the failure of Gödel to clarify the principal assertions involved in his suggested proof outline, where there is a reliance on intuition rather than logical transparency.

The result of this vagueness of presentation is that there is no clear delineation of the meta-language and the sub-languages involved, with the result that a crucially erroneous equivalence is asserted between an expression of the meta-language and a sub-language, an equivalence which is logically untenable. It is shown here that the self-reference generated by Proposition VI in Gödel's proof relies on this erroneous intuitive assumption, and hence the self-reference of that Proposition is logically untenable, and there is no logical basis for Gödel's result.

This paper uses straightforward logic and does not rely on any philosophical or semantical arguments.

2 Contents

Current Revision: V6-16022022

Revision History
V6-16022022

- Changed the name of the formal system name from **F** to **P**.
- In response to requests for a brief summary of the flaw in Gödel's paper, added section 5A The Crucial Erroneous Assumption which demonstrates the irremediable error in an erroneous assumption by Gödel and upon which his result is totally reliant.
- Added note at beginning of section 7.8 re the addition of the section 5A.
- End of section 7.9 removed as now covered by section 5A.

V5-15032020

- Added section "The impossibility of Gödel's result" in response to suggestions that the assumed equivalence of the Z function and the Gödel numbering function is immaterial to Gödel's result.
- Added a summarizing note to the preface explaining why this paper is necessarily lengthy.

V4-170814

- Improved formatting.
- Minor typographical and presentational alterations.
- Simplified Proof in appendix revised.

V3-050109

- Initial note re the book 'The Shackles of Conviction' removed.
- Additional section (section 6) added, which demonstrates the flaw in Gödel's proof by a simpler method. This section was added since some readers found the demonstration in the previous version too difficult. However, the more detailed analysis of the previous version is more instructive as it shows how Gödel's proof is just one instance of confusion over the use of what is called 'higher-order' logic, and so that analysis is still present in this version, in Section 7.
- The section on meta-language and sub-language, (Section 5 in this version) has been expanded.
- Most of the preface has been removed, as it was still being construed as part of the main argument of the paper, despite a note to the contrary.
- As a result of the above change, there are changes in the numbering of the paper and some minor textual changes.

V2-240608 - There were two areas of alteration in this revision.

- The term "Gödel's Incompleteness Theorem" is commonly used to refer to Gödel's actual proof as well as one the main propositions of his paper. This has being seized on by various persons choosing that the term "Gödel's Incompleteness Theorem" means the main proposition of his paper where it was clearly evident that the intention was that it should mean Gödel's actual proof. That context dependency of the term "Gödel's Incompleteness Theorem" has been removed.
- While I had originally not added a note that my preface was not to be taken as part of the main argument of my paper, deciding that it was so obvious that it would be pedantic to do so. Apparently not. There are no other changes to the text apart from the above.

V1-210608 - Minor alterations in formatting and presentation.

3 Preface

This paper only deals with Gödel's proof as given in his paper 'On Formally Undecidable Propositions of Principia Mathematica and Related Systems',[1] and does not claim to apply to proofs of other Incompleteness Theorems. This paper refers only to Gödel's original proof, and the argument presented in this paper relates to Part 2 of Gödel's paper. This paper does not deal with other proofs that are asserted to be 'versions' of Gödel's proof.

Because Gödel did not give a proof of his Proposition V, and merely gave an outline sketch of how such a proof might proceed, and remarked that such a proof would be lengthy, the content of this paper is necessarily lengthy as the proof has to be constructed here in detail.

When this is done, an analysis demonstrates that Gödel's perfunctory outline relies on several logically invalid assumptions which result in a conflation of meta-language and sub-language. For a proof which is asserted to be a proof in one language regarding another language, such assumptions are not inconsequential; they are fundamental errors that render Gödel's result impossible.

In particular, Gödel assumes an equivalence between his Z function (his relation 17) and his numbering function, for numerical values of the free variable, even though their domains are not identical, and it is shown in Section 5A that this claim of equivalence is erroneous. The detailed analysis shows that Gödel's meta-language in which he defines his numbering function is a meta-language both to the formal system **P** and to the language of number-theoretic relations, and his numbering function is in a language that is a meta-language to the object language to which his Z function belongs, and his assumption of equivalence is a logical error that conflates levels of language.

4 Introduction

4.1 Layout of this Paper

This section is an introductory section, which also covers some of the basic concepts involved in Gödel's proof. Section 5 is an overview of some of the basic logical principles involved in the distinction between a meta-language and a sub-language. Section 6 is a consideration of Gödel's Proposition V according to the principles of section 5 which demonstrates the fundamental flaw in Gödel's proof. Section 7 is a detailed examination of Gödel's proof of his Proposition V from the viewpoint of the principles outlined in section 5, and which also demonstrates the fundamental flaw in Gödel's proof. It should be noted that the sections 6 and 7 are merely different ways of demonstrating the same fundamental flaw.

4.2 Symbols and terms used in this paper

0	$\equiv$	zero	$\forall$	$\equiv$	for all
f	$\equiv$	the successor of	$\Rightarrow$	$\equiv$	implies
$\neg$	$\equiv$	not	$\Leftrightarrow$	$\equiv$	correspondence
$\vee$	$\equiv$	or	$\wedge$	$\equiv$	and
$\exists$	$\equiv$	there exists	$\equiv$	$\equiv$	equivalence

4

Various relations are referred to and explained in the main text. The following is only a summary of the principal relations referred to

$\Phi(X)$ The function that describes what is commonly referred to
as the Gödel numbering system, see section 4.3

$\Psi(X)$ A function that gives a specific natural number of the
meta-language for every symbol of the formal system **P**, see section 4.3

R, $R(x)$ A relation of natural numbers

FRM^P A formula of the formal system **P**

PRF^P A proof scheme of the formal system **P** that is
a proof of some formula

$x\,Proof^P\,y$ The symbol combination x is a proof in the system **P**
of the symbol combination y

T^A A mapping function

$0, f0, ff0, fff0, \ldots$ Symbol combinations for natural numbers

$Fr(x,y)$ x is a free variable in the symbol sequence y

The following are relations that are defined in Gödel's paper that will be referred to in the text. The version of Gödel's paper referred to here is the English translation of Gödel's paper by B.Meltzer.[1] There is also a translation by Martin Hirzel[2] and the corresponding terms are indicated below.

$Subst(a,v,b)$ Gödel's meta-mathematical notion of substitution, see section 5, page 12
(also in the format of $Subst\,a\binom{v}{b}$ in some versions of Meltzer's translation,
and as $subst\,a\binom{v}{b}$ in Hirzel's translation)

$Z(y)$ Gödel's relation 17   ($number(y)$ in Hirzel's translation)

$Sb(x,v,w)$ Gödel's relation 31
(also in the format of $Sb\left(x\,\genfrac{}{}{0pt}{}{v}{y}\right)$ in some versions of Meltzer's translation,
and as $subst(x,v,w)$ in Hirzel's translation)

xBy Gödel's relation 45   ($proofFor(x,y)$ in Hirzel's translation)

$Bew(x)$ Gödel's relation 46   ($provable(x)$ in Hirzel's translation)

Note that the definition of the terms 'recursive' and 'ω-consistent' are not here defined, since these terms are defined in Gödel's paper, and their precise definition is immaterial to the argument here presented.

4.3 The Gödel numbering system - a brief overview

In a formal system, the basic symbols of the system are simply placed one after another in a particular order to create a formula of the system. In his paper, Gödel defines a relationship between symbol sequences of a formal system $\mathbf{P}$ and numbers. This is referred to as a one-to-one correspondence. The system that Gödel defined to establish this relationship between a sequence of symbols of his formal system $\mathbf{P}$ and a number is basically as follows:

First, Gödel defines a relationship between each symbol of the formal system $\mathbf{P}$ that is not a variable to an associated number as below.

$$
\begin{array}{llllllll}
\mathbf{0} & \Leftrightarrow & \mathbf{1} & \quad \neg & \Leftrightarrow & \mathbf{5} & \quad \forall & \Leftrightarrow & \mathbf{9} & \quad) & \Leftrightarrow & \mathbf{13} \\
f & \Leftrightarrow & \mathbf{3} & \quad \vee & \Leftrightarrow & \mathbf{7} & \quad (& \Leftrightarrow & \mathbf{11}
\end{array}
$$

Each variable of the first type of the formal system $\mathbf{P}$ (with the domain of individuals of the formal system, that is, entities of the form $\mathbf{0}, f\mathbf{0}, ff\mathbf{0}, fff\mathbf{0}, \dots$) is defined as corresponding to a prime number greater than $\mathbf{13}$, i.e., $\mathbf{17, 19, 23, \dots}$. For convenience we shall refer to these variables of the formal system $\mathbf{P}$ as $v_1, v_2, v_3, \dots$.[1]

The above relationship is a one-to-one correspondence, that is, a bijective function; we designate this as $\mathbf{\Psi}(t)$.[2] The value of $\mathbf{\Psi}(t)$ is the number that corresponds to the symbol given by t, that is, it returns one of the values $\mathbf{1, 3, 5, 7, 9, 11, 13, 17, 19, 23, \dots}$ etc.

Gödel defines the relationship of a natural number to a symbol sequence of the formal system $\mathbf{P}$ as follows:

The symbol at the nth position is represented by the nth prime number to the power of the number corresponding to that symbol. This method of assigning a number to a sequence of formal symbols is also a one-to-one correspondence, i.e., a bijective function. This function is designated as $\mathbf{\Phi}(X)$, where X is a sequence of formal symbols. The formal symbol sequence X may be of any finite length, and the function $\mathbf{\Phi}(X)$ references by the function $\mathbf{\Psi}(t)$ every symbol of the symbol sequence X.

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[1] The details of the symbols for variables of the second type or higher are irrelevant to the argument presented in this paper and are not considered here.

[2] While Gödel does not explicitly define a function with the designation $\mathbf{\Psi}(t)$, his definition of the numbering system implicitly defines this function.

5 Meta-language and Sub-language

5.1 Preliminary Notes

- *Sub-language:* In the rest of this paper, if the symbols for variables of a language are not symbols for variables in the meta-language, then that language is referred to as a sub-language to the meta-language.

- In the rest of this paper, capital symbols are used to represent variables of the meta-language, to make it easier to distinguish symbols that are variables of the meta-language and symbols that are variables of a sub-language such as the formal system **P**.

- In this paper, for convenience, the meta-language that is the proof language in which Gödel's Proposition V and its proof are stated is called the language **PV**.

5.2 Language, Variables and Specific Values

Gödel's paper involves the notion of a meta-language. In his paper the proof language is considered to be a meta-language and the formal system to be referenced by that meta-language. The formal system **P** is considered to be simply a collection of specific symbols: 0, f, $\neg$, $\forall$, $\vee$, $($, $)$, together with the symbols for variables. The symbols for variables of the formal system **P** are not symbols for variables in the meta-language, but, as for the other symbols, they are only specific values in the meta-language - that is, the rules of syntax that apply to variables of the meta-language do not apply to the symbols for variables of the formal language. For the moment we ignore the implications where the same symbols are used for relational operators in the meta-language and the formal system **P** (for example, '$\forall$' might be used as the universal quantifier symbol in both the formal language **P** and the meta-language).

We first look at the elementary concepts of propositions and variables referred to by the propositions of Gödel's proof. In Gödel's paper, as is the norm, the concept of *'variable'* is taken for granted, without any need envisaged for clarification or definition of what is meant by *'variable'*. But the existence of variables implies that there are two types of entity referenced by relational operators: variables and non-variables.

We note that the domain of any variable cannot include that variable itself - the domain of a variable is a domain of values that are not variables in that particular language. For any proposition with a quantifier on a variable, either the universal quantifier (such as *'For all. . . '*), or the existential quantifier (such as *'There exists some. . . '*), that proposition can only imply a proposition where the variable is substituted by a specific value of the domain of that variable. For example, the proposition $\forall x(x > 3)$ might imply $4 > 3, 5 > 3$, etc but it cannot imply $x > 3$ since $x > 3$ is not a proposition because it contains a free variable. Nor can it imply, for example, $y > 3$, where y is also a variable of the language being used.

Gödel's Proposition V is a proposition that relies on other propositions for its proof. Being propositions, none of these propositions can have any free variables (that is, free variables of the meta-language **PV**).

A clear distinction between variables of these propositions and the specific values that constitute their domains must apply to all those propositions. And since all those propositions all have to be in the same logical language (the language **PV**), there must be a clear distinction between the variables of the language **PV** and the specific values of that language **PV**.

5.3 Symbols for Meta-language and Sub-language

As well as the symbols for a meta-language and a sub-language necessarily being distinct, a meta-language can always be chosen to have different symbols for relational operators from the sub-languages that it refers to; there is no logical reason that stipulates that such symbols must be identical for a sub-language and a meta-language. It follows that if an analytical result is necessarily dependent on the same symbol being used for a relational operator in the meta-language and a sub-language, then that result is dependent on confusion between the meta-language and the sub-language, and that such a result cannot have any logical validity.

The same also applies to the specific values that constitute the domains of the variables of the meta-language and the sub-language - different symbols can always be chosen for the specific values of the meta-language and for any sub-language. It follows that there is no logical requirement for the symbols for numbers in the formal system **P** (of the form 0, $f0$, $ff0$, $fff0$, $\ldots$) to be also symbol combinations for numbers in the meta-language. Note that while the meta-language needs to be able to refer to the symbol combinations that constitute the numbers of the formal system **P** (of the form 0, $f0$, $ff0$, $fff0$, $\ldots$), for the meta-language, those are simply symbol combinations with no meaning in the meta-language; simply because the symbols for the numbers of the formal system **P** represent the same abstract concept as the symbols used for the numbers of the meta-language provides no logical justification for using the same symbols for that concept in the meta-language and in the sub-language. However, since the above might be subject to some controversy, this aspect of the meta-language and the sub-language will not be used here to justify the argument presented in this paper.

5A The Crucial Erroneous Assumption

This section has been added to provide a brief description of the error in a crucial assumptive assertion in Gödel's paper, which is easily shown to be false. The entire argument of Gödel's proof relies completely on the assertion, and the remainder of this paper demonstrates in detail why this is the case.

Relation 17 of Gödel's paper states that:

$$Z(n) = nN[R(1)]$$
$Z(n)$ is the *number-string* for the number n [3]

Gödel previously defined that an italicized word (such as *number-string* above) refers to the number calculated by his numbering function Φ for a given sequence of symbols of the formal system **P** (see

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[3] "$Z(n)$ ist das *Zahlzeichen* für die Zahl n" in the original German.

section 4.3). When he asserts that "$Z(n)$ is the *number-string* ..." this is an assumptive assertion of an equivalence of this Z function and his numbering function Φ, such that their calculated values are equal for the same value of their free variables, and the result of his paper has a complete reliance on this assumption.

The assertion gives rise to the question of how such an equivalence can apply if the formats of the numbers that constitute the domains of the free variables of Z function and his numbering function Φ are different. However, we can consider for a moment the case where the format for numbers of the domain of the free variable of the Z function is precisely identical to that of the formal system $\mathbf{P}$ of Gödel's paper, which gives the assertion of equivalence as applying for the domain of the free variable n as natural numbers in that format: $Z(n) = \Phi(n)$.

However, even the circumvention of the issue of different formats in this way still does not achieve an equivalence of the functions. This is because there can be infinitely many expressions that have the same value as any given number, and which are also valid values for the substitution of the free variable of the Z function. For example, using the format of the formal system $\mathbf{P}$, we have that $ff0 + fff0 = fffff0$, and both $ff0 + fff0$ and $fffff0$ are valid substitutions for the free variable of Z. But the expressions "$ff0 + fff0$" and "$fffff0$" are not the same value for the domain of the free variable of the Φ numbering function, since by definition, this function must *always* give a different value for *every* expression of the formal system $\mathbf{P}$, regardless of the computed value of that expression.

In particular, the assertion of equivalence of the Z and Φ functions is a requirement of the argument of Gödel's Proposition VI. That being the case, one cannot assert equivalence and then later ignore the implications of that assertion; the asserted equivalence must hold throughout the argument. In Proposition VI the free variable of the Z function is substituted by the function $\mathbf{17\ Gen\ q}$, which is not an expression that is expressed by a sequence $fff\ldots 0$, although it does have a certain numerical value when substituted for the free variable of the Z function. But even if we suppose that one might substitute the free variable of the Φ function by a formal expression that expresses the syntactical content of $\mathbf{17\ Gen\ q}$, that expression does *not* have a numerical value when substituted for the free variable of the Φ function, since substitutions of its free variable are considered only as sequences of symbols without *any* consideration of their syntactical value in the formal system. Hence the resulting calculated value of the Φ function is not identical to the resulting value of the Z function, which contradicts the prior assertion of the equivalence of the Z and Φ functions.[4]

Hence, even with ignoring issues that might arise from different number formats, Gödel's intuitive assertion of equivalence of the Z function and the Φ function is logically untenable and has no validity whatsoever. Since Gödel's result relies completely on this erroneous assumption, the result of Gödel's proof is untenable and the proof is fundamentally flawed. The remainder of this paper demonstrates in detail the confusion of levels of language engendered by this erroneous assumption.

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[4] A plea might be made that $\mathbf{17\ Gen\ q}$ must have some numerical value which would be a valid substitution for the free variable in both the Z and the Φ function. However, one cannot make that plea in isolation, and in fact the plea actually is $\exists n, Z(n) = \Phi(n) \wedge n = \mathbf{17\ Gen\ q}$. But if $n = \mathbf{17\ Gen\ q}$, that equality holds everywhere and so $\mathbf{17\ Gen\ q}$ should be a valid substitution for n wherever n appears, but this cannot be the case.

6 Gödel's Proposition V

6.1 Meta-Language and 'Number-Theoretic Relations'

Gödel's Proposition V essentially states:

> *'For every recursive number-theoretic relation for which there exists n free variables,*
>
> *there exists a corresponding number r, and for every x, ... '*

If that is a proposition, then that is a proposition with no free variables (of the language **PV** in which it is stated) and all of its variables are bound variables. One bound variable is *'recursive number-theoretic relation'*, which is bound by the quantifier *'For every ... '* and another is *'number'* (... *a corresponding number r*), which is bound by the quantifier *'there exists ... '*.

And when the above refers to *'n free variables'* of the recursive *'number-theoretic relation'*, the word *'variables'* in that proposition is itself a variable. It is a variable in the language **PV** of the proposition because it is not a specific value. It represents a variable of a *'number-theoretic relation'* . And it is a bound variable, because it is bound by *'there exists ... '* Since the word *'variables'* is actually a variable in the language **PV** of Gödel's Proposition V, then there can also be propositions in that same language **PV** which reference some specific value that may be represented by that variable. Such a specific value is a variable of a *'number-theoretic relation'* , and it follows that the value cannot be itself a variable in that language **PV**.

This means that the language **PV** of Gödel's Proposition V is a language in which *'number-theoretic relations'* are specific values, and in which the variables of *'number-theoretic relations'* are specific values. It is also the case that in the language **PV** of Gödel's Proposition V, formal formulas are also specific values, and the variables of formal formulas are specific values. That means that the language **PV** of Gödel's Proposition V is a meta-language to both the formal language and to *'number-theoretic relations'* .

6.2 Analysis of Proposition V in terms of meta-language and sub-language

For the purposes of simplification, in this section of this paper we will deal with Gödel's Proposition V

- for relations of only one free variable,

- without consideration of the negation of the *'number-theoretic relation'* , and

- without interpretative descriptions

since the argument here presented does not rely on these aspects of the proposition.

Gödel's Proposition V is thus given as:

(6.2.1) *For all recursive 'number-theoretic relations'* $R(X)$*, for all* X*,*

(6.2.2)   *there exist numbers* Q*,* P *and* U*, where*

        P *and* U *are prime,* $U > 13$*, and* P^U *is a factor of* Q*, and*

(6.2.3) $R(X) \Rightarrow Bew\{Sb[Q, U, Z(X)]\}$

Note that in the above, in order to distinguish variables of the meta-language **PV** from variables of the sub-language, the variables of the meta-language are in capital letters. It should be noted that Gödel's Proposition V is of a form that is commonly referred to as a 'higher-order logic' expression. However, this of itself does not confer logical validity to the expression (various anomalies commonly exist in such expressions but are usually ignored - this is dealt with in detail in Section 7.6).

Now, in the above, R (or $R(X)$) is a variable of the language **PV**. This means that for any specific value of R, that is, for any specific *'number-theoretic relation'* , if Gödel's Proposition V is a valid proposition, then that proposition must imply, for the specific value R_1 (here we designate one such *'number-theoretic relation'* as R_1):

(6.2.4) *Given the specific recursive 'number-theoretic relation'* $R_1(X)$*, for all* X*,*

(6.2.5)   *there exist numbers* Q*,* P *and* U*, where*

        P *and* U *are prime,* $U > 13$*, and* P^U *is a factor of* Q*, and*

(6.2.6) $R_1(X) \Rightarrow Bew\{Sb[Q, U, Z(X)]\}$

Now, *'number-theoretic relations'* are simply specific values of the meta-language **PV**. Conventionally, an expression such as '$R_1(X)$' refers to an expression in which the symbol X is the free variable of the expression R_1. In the above, however, this cannot be the case, since X is a variable of the meta-language, whereas R_1 is a specific value of the language **PV** - a specific expression of the sub-language of *'number-theoretic relations'* . X cannot be a variable of R_1, since it is a variable of the meta-language **PV**.[5]

It follows that for '$R_1(X)$' to be a valid expression of the language **PV**, the intended evaluation of $R_1(X)$ in the language **PV** is clearly:

        '*the number-theoretic relation that is given when the free variable*

            of the number-theoretic relation R_1 *is substituted by* X*.*'

This evaluation of '$R_1(X)$' refers to a *'number-theoretic expression'*, which is a specific value of the meta-language.

We note that this follows the way that Gödel refers to substitution of the free variable of a formula of the formal system **P** (formulas of the formal system **P** also belong to a sub-language) – he refers to the

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[5] While in many cases of mathematical analysis, such distinctions between meta-language and sub-language are ignored without any apparent problems arising, this can never imply that such distinctions can always be ignored.

concept '$Subst(F, v, n)$', where F stands for a formula, v a variable and n a specific value, and Gödel intends that this represents:

'*the formula derived from F, when the variable v in the formula F,*

wherever it is free, is substituted by n.'

In a meta-language that must distinguish between symbol combinations of the formal system **P** and other expressions, there cannot be a mathematical equality between the expression 'the formula derived from F, when the variable v in the formula F, wherever it is free, is substituted by n' and that one particular formal system formula, nor between $Subst(F, v, n)$ and that one particular formal system formula.[6] This follows since their properties, within the meta-language **PV**, are not identical – the expression $Subst(F, v, n)$ is not itself actually an expression of the formal sub-language, whereas the actual formula that it refers to is an expression of the formal sub-language.

In order to avoid confusion, we can use Gödel's unambiguous terminology $Subst(R_1, fv, X)$, where fv is the free variable of R_1, instead of the ambiguous terminology $R_1(X)$.

As it stands, this would give Proposition V, for a specific *'number-theoretic relation'* R_1 as:

(6.2.7) *Given the specific recursive 'number-theoretic relation' $R_1(X)$, for all X,*

(6.2.8)   *there exist numbers Q, P and U, where*

      P and U are prime, $U > 13$, and P^U is a factor of Q, and

(6.2.9) $Subst(R_1, fv, X) \Rightarrow Bew\{Sb[Q, U, Z(X)]\}$

However, in the meta-language, specific values, such as that given by $Subst(R_1, fv, X)$ for a given value of X, cannot imply anything.[7]

Of course, in Gödel's Proposition V, besides '$R_1(X)$' representing the concept $Subst(R_1, fv, X)$ where it occurs in (6.2.3) and (6.2.6) above, the expression '$R_1(X)$' is intended to represent the concept that $Subst(R_1, fv, X)$ is provable by the use of the axioms of the formal system **P**. This is confirmed by Gödel, where he states in a footnote to his outline proof of Proposition V that:

'*Proposition V naturally is based on the fact that for any recursive relation R, it is decidable … from the axioms of the [formal system], whether the relation R holds or not.*'

To remove the ambiguity, we denote the intended concept represented by $R_1(X)$ by $Provable[Subst(R_1, fv, X)]$.[8]

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[6] Gödel notes in his paper that '$Subst$' is a meta-mathematical concept rather than a mathematical concept.

[7] Even if it were supposed that a specific value might imply that same specific value, that would mean that, for any given value of X, either the specific value given by $Subst(R_1, fv, X)$ is the same specific value as $Bew\{Sb[Q, U, Z(X)]\}$, or that the specific value given by $Subst(\neg R_1, fv, X)$ is the same specific value as $Bew\{Neg \, Sb[Q, U, Z(X)]\}$. That, of course, would be absurd, since these specific values are specific *'number-theoretic relations'* and each individual *'number-theoretic relation'* is a different expression of the sub-language of *'number-theoretic relations'* .

[8] Or we might denote it by $True[Subst(R_1, fv, X)]$ since in this context 'True' and 'Provable' are identical concepts - see Appendix 1: Provability and Truth for the reason why this is the case.

As it stands, this would give for Proposition V, for a specific *'number-theoretic relation'* R_1:

(6.2.10) *Given the recursive 'number-theoretic relation'* $R_1(X)$, *for all* X **:**

(6.2.11)   *there exist numbers* Q, P *and* U, *where*

        P *and* U *are prime,* $U > 13$, *and* P^U *is a factor of* Q, *and*

(6.2.12) $Provable[Subst(R_1, fv, X)] \Rightarrow Bew\{Sb[Q, U, Z(X)]\}$

At this point, consider Gödel's outline proof of Proposition V, which is:[9]

(6.2.13) For the *'number-theoretic relation'* R_1 there is a corresponding formal formula F_1,

        and $Subst(R_1, fv, X)$ corresponds to $Subst(F_1, fv, X)$

(6.2.14) For the formal formula F_1, there is a corresponding Gödel number Q_1,

        and $Subst(F_1, fv, X)$ corresponds to $Sb[Q_1, U, Z(X)]$

(6.2.15) $Provable[Subst(F_1, fv, X)]$ corresponds to $Provable[Subst(R_1, fv, X)]$

(6.2.16) $Bew\{Sb[Q_1, U, Z(X)]\}$ corresponds to $Provable[Subst(F_1, fv, X)]$

We may put this as follows, using $\Leftrightarrow$ as indicating a one-to-one correspondence:

(6.2.17) $$R_1 \Leftrightarrow F_1$$

(6.2.18) $$Subst(R_1, fv, X) \Leftrightarrow Subst(F_1, fv, X)$$

(6.2.19) $$F_1 \Leftrightarrow Q_1$$

(6.2.20) $$Subst(F_1, fv, X) \Leftrightarrow Sb[Q_1, U, Z(X)]$$

(6.2.21) $$Provable[Subst(R_1, fv, X)] \Leftrightarrow Provable[Subst(F_1, fv, X)]$$

(6.2.22) $$Provable[Subst(F_1, fv, X)] \Leftrightarrow Bew\{Sb[Q_1, U, Z(X)]\}$$

From (6.2.21) and (6.2.22) we have:

(6.2.23) $$Provable[Subst(R_1, fv, X)] \Leftrightarrow Bew\{Sb[Q_1, U, Z(X)]\}$$

which is the basis of Gödel's outline proof of his Proposition V.

Up to this point we have not considered the expression $Bew \dots$ as it appears in Proposition V.

But here there is an insurmountable problem. For, on the one hand, the expression $Bew \dots$ is necessarily a *'number-theoretic relation'*.[10] However, as shown above for the relation R, the expression $Bew\{Sb[Q, U, Z(X)]\}$ cannot be a *'number-theoretic relation'*, since it includes the variables Q, U, and X, which are variables of the meta-language **PV**.

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[9] $Provable[Subst(F_1, fv, X)]$ indicates that the formula given by $Subst(F_1, fv, X)$ is provable in the formal system **P**.

[10] For the purposes of Gödel's proof, Bew must be a *'number-theoretic relation'*, since:
  a) $Bew\{Sb[a, b, Z(x)]\}$ is Gödel's *'number-theoretic relation'* 46, derived from his relation 45, $y \, B \, Sb[a, b, Z(x)]$ by the addition of the existential quantifier $\exists$, so that $Bew\{Sb[a, b, Z(x)]\} \equiv \exists y, y \, B \, Sb[a, b, Z(x)]$, and it follows that the expressions are in the same language, and
  b) to complete Gödel's proof, in his Proposition VI the expression B is a *'number-theoretic relation'*, and Bew and B are expressions in the same language.

Even if we allow that for any specific values of Q, U, and X, that the resultant expressions for $Bew\{Sb[Q, U, Z(X)]\}$ are *'number-theoretic relations'*, then, since they are *'number-theoretic relations'* they are simply specific values in the meta-language **PV** - they are not propositions in the language **PV**.[11]

But the left-hand side of (6.2.23), for any specific values of the variable X, is a proposition of the language **PV** – it is an expression that is not a specific value.

It is apparent that a logical analysis of Gödel's proof of his Proposition V demonstrates that it leads to a logical absurdity where a proposition is asserted to imply a specific value. It shows that Gödel's Proposition V is not a valid proposition at all – it is an expression that confuses sub-language and meta-language, rendering its purported outline proof completely untenable.

In a proof that is supposedly a proof by a meta-language regarding a sub-language, the distinction between meta-language and sub-languages should be the paramount consideration, but Gödel's outline proof completely fails to make a clear distinction between sub-language and meta-language.

Note that if different relational operators are used for the sub-language of *'number-theoretic relations'* and the meta-language, the confusion regarding the expressions $Bew\{Sb[Q_1, U, Z(X)]\}$ can readily be seen. For in the definition of Bew, that definition would define Bew as either an expression of a sub-language or of the meta-language; any further references to the expression could not evade the distinction between meta-language and sub-language.

It might be supposed that the problem can be overcome by applying the notion of $Provable$ to the right hand side of the expression giving:

(6.2.24) $Provable[Subst(R_1, fv, X)] \Leftrightarrow Provable[Bew\{Sb[Q_1, U, Z(X)]\}]$

This is not the case, since the outline proof that Gödel gives for Proposition V defines that $Bew\{Sb[Q_1, U, Z(X)]\}$ corresponds to $Provable[Subst(R_1, fv, X)]$. There is no basis whatsoever to support the notion that $Provable[Bew\{Sb[Q_1, U, Z(X)]\}]$ is implied by $Provable[Subst(R_1, fv, X)]$. The outline proof that Gödel gives for Proposition V can only give the result of (6.2.23), a result which is a logical absurdity.

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[11] The same would be the case if we apply the notion of $Subst$ to the expression $Bew\{Sb[Q, U, Z(X)]\}$, as was done for the relation R.

7 Detailed Overview of Gödel's Proposition V

While the previous sections are sufficient to demonstrate the logical flaw in Gödel's Proposition V, it is instructive to follow, as logically as possible, Gödel's outline proof of his Proposition V. This is done in the following sections.

7.1 Sub-languages and Mapping Functions

Inherent in Gödel's Proposition V is the notion of mapping from one system to another system. It is instructive to consider a simple mapping of one formal system **A** to another formal system **B**, since we shall be using certain terms later in the text that are explained here.

In simple terms, such a mapping might be described by:

(7.1.1) $\forall FRM^A,$

(7.1.2) $\exists PRF^A, \{PRF^A\, Proof^A\, FRM^A\} \Rightarrow \exists FRM^B, \exists PRF^B, \{PRF^B\, Proof^B\, FRM^B\}$

where PRF^A and PRF^B, FRM^A and FRM^B are variables of the meta-language – their domain is symbol combinations of the formal languages **A** and **B** respectively (which may be proof schemes or formulas), and $Proof^A$ is a relation, where $d\, Proof^A\, e$ means that the symbol combination d is a proof in the language **A** of the symbol combination e, and similarly for $Proof^B$.

The expression may be interpreted as:

*'For every formula of the formal system **A**, if there exists a formal proof scheme in the system **A** which is the proof of that formula, that implies that there exists a corresponding formula in the formal system **B**, for which there exists a formal proof scheme in the formal system **B**.'*

The proposition is a proposition in a meta-language where both the formal system **A** and the formal system **B** are sub-languages to the meta-language, and where the variables of these formal languages are not variables of the meta-language. The expressions (7.1.1) – (7.1.2) are expressions of the meta-language regarding entities of the sub-languages **A** and **B**.

The basis for the assertion of the proposition is that there exists a mapping function T^{A-B} in the language of the proposition, and this mapping function asserts that for a given formula of the system **A**, there exists a formula of the system **B**. This gives:

(7.1.3) $\forall FRM^A, \exists FRM^B, \{FRM^B = T^{A-B}(FRM^A))\},$

(7.1.4) $\quad\quad \exists PRF^A, \{PRF^A\, Proof^A\, FRM^A\}$

(7.1.5) $\quad\quad\quad \Rightarrow \exists [T^{A-B}(PRF^A)], \{[T^{A-B}(PRF^A)]\, Proof^B\, [T^{A-B}(FRM^A)]\}$

Or, designating $T^{A-B}(PRF^A)$ by PRF^B, and $T^{A-B}(FRM^A)$ by FRM^B this gives:

(7.1.6) $\forall FRM^A, \exists FRM^B, \{FRM^B = T^{A-B}(FRM^A)\},$

(7.1.7) $\qquad \exists PRF^A, \{PRF^A \, Proof^A \, FRM^A\} \Rightarrow \exists PRF^B, \{PRF^B \, Proof^B \, FRM^B\}$

 which is similar to the expression given by (7.1.1) – (7.1.2) above.[12]

As noted previously, Gödel uses the concept of $\textbf{\textit{Subst}}$ in his proof: 'By $\textbf{\textit{Subst}}(\textbf{\textit{a}}, \textbf{\textit{v}}, \textbf{\textit{b}})$ (where $\textbf{\textit{a}}$ stands for a formula, $\textbf{\textit{v}}$ a variable and $\textbf{\textit{b}}$ a sign of the same type as $\textbf{\textit{v}}$) we understand the formula derived from $\textbf{\textit{a}}$, when we replace $\textbf{\textit{v}}$ in it, wherever it is free, by $\textbf{\textit{b}}$.' Applying this to the above gives:

(7.1.8) $\forall FRM^A, \forall W^A, \forall X^A, \exists FRM^B,$

$\qquad \{FRM^B = T^{A-B}(FRM^A)\}, \exists W^B, \{W^B = T^{A-B}(W^A)\}, \exists X^B, \{X^B = T^{A-B}(X^A)\},$

(7.1.9) $\exists PRF^A, \{PRF^A \, Proof^A \, Subst(FRM^A, W^A, X^A)\}$

$\Rightarrow \exists [T^{A-B}(PRF^A)], \{[T^{A-B}(PRF^A)] \, Proof^B \, Subst[T^{A-B}(FRM^A), T^{A-B}(W^A), T^{A-B}(X^A)]\}$

(7.1.10) $\Rightarrow \exists PRF^B, \{PRF^B \, Proof^B \, Subst(FRM^B, W^B, X^B)\}$

where the expression $\textbf{\textit{Subst}}$ is an expression of the meta-language, and

W^A, W^B, X^A and X^B are variables of the meta-language, where
W^A is a variable whose domain is variable symbols of the formal system **A**,
W^B is a variable whose domain is variable symbols of the formal system **B**,
X^A is a variable whose domain is number symbols of the formal system **A**,
X^B is a variable whose domain is number symbols of the formal system **B**,
and where the mapping function T^{A-B} applies to give $T^{A-B}(PRF^A) = PRF^B$.

This expression is somewhat cumbersome, which is the reason such expressions are commonly reduced to a simpler form, such as:

(7.1.11) $\forall FRM^A, \forall W^A, \forall X^A,$

(7.1.12) $\qquad \exists PRF^A, \{PRF^A \, Proof^A \, Subst(FRM^A, W^A, X^A)\}$

(7.1.13) $\qquad\qquad \Rightarrow \exists PRF^B, \{PRF^B \, Proof^B \, Subst(FRM^B, W^B, X^B)\}$

where it is implied that the mapping function T^{A-B} applies to give $T^{A-B}(PRF^A) = PRF^B$,
$T^{A-B}(FRM^A) = FRM^B, T^{A-B}(X^A) = X^B$, and $T^{A-B}(W^A) = W^B$.

This expression will suffice provided that the underlying implicit meta-mathematical assumptions are not ignored.

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[12] Note: the mapping as used here is a meta-mathematical concept, rather than a strict equality, since in a meta-language that must distinguish between symbol combinations of the formal system and other expressions, there is not a mathematical equality between FRM^B and $T^{A-B}(FRM^A)$ since $T^{A-B}(FRM^A)$, while representing a formula of the formal system **B** is not itself a formula of the formal system **B**, and does not have the properties of a formula of the formal system **B**. This strict distinction should be borne in mind.

7.2 Meta-language and Sub-language

In the mapping of an expression from some sub-language **Ln** to another sub-language **Lm**, we expect that the definition for a mapping will define the correspondence of the variables of the language **Ln** to the variables of the language **Lm**, which for any given mapping is a one-to-one correspondence. Since there are infinitely many such variables for any given language **Ln**, the definition of a general correspondence that is applicable to all relations must be by an expression that includes a variable that has the domain of all variables of the language **Ln**, and a variable that has the domain of all variables of the language **Lm**.

Thus in general, we have some mapping T^{Ln-Lm}, so that $X^{Ln} = T^{Ln-Lm}(Y^{Lm})$[13] is an expression where X^{Ln} is a variable of the meta-language with the domain of variables of the language **Ln**, and Y^{Lm} is a variable of the meta-language with the domain of variables of the language **Lm**. Clearly, in such an expression, the variables of the language **Ln** and of the language **Lm** are not variable quantities of the meta-language, but are specific values of the meta-language. That is, for some particular variable of the language **Ln**, that we call v^{Ln}, there is some particular variable of the formal language **Lm** that we call v^{Lm}, and we can assert that $v^{Ln} = T^{Ln-Lm}(v^{Lm})$. Since v^{Lm} is not a variable in the meta-language, and since $v^{Ln} = T^{Ln-Lm}(v^{Lm})$, it follows that v^{Ln} also is not a variable in the meta-language.

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[13] Again, for a meta-language that must distinguish between symbol combinations of sub-language and other expressions, such a mapping is a meta-mathematical concept, rather than a strict mathematical equality; this strict distinction should be borne in mind.

7.3 The Outline Proof of Proposition V

For the purposes of simplification, in this section of this paper, as in Section 6, we will deal with Gödel's Proposition V

- for relations of only one free variable,

- without consideration of the negation of the *'number-theoretic relation'*, and

- without interpretative descriptions

since the argument here presented does not rely on these aspects of the proposition.

The assertion of Proposition V is thus given as:

(7.3.1) *For all recursive 'number-theoretic relations' $R(X)$, for all X,*

$\qquad$ *there exist numbers Q, P and U, where*

$\qquad$ *P and U are prime, $U > 13$, and P^U is a factor of Q, and*

(7.3.2) $R(X) \Rightarrow Bew\{Sb[Q, U, Z(X)]\}$

The implied proof of Proposition V can be considered to consist of four principal assertions:

Assertion I: the expression *'there exists an expression A in the formal system P that is a proof of the formal system formula B where its free variable has been substituted by some number'* has a corresponding definable *'number-theoretic relation'*.

Assertion II: for any recursive *'number-theoretic relation'*, then for any specific values substituted for the free variables of the relation, it is decidable by finite means whether that relation or its negation holds.

Assertion III: for every recursive *'number-theoretic relation'* R, there is a corresponding formal system formula FRM^P. The formal system formula will have the same number of free variables as the *'number-theoretic relation'*.

Assertion IV: if a recursive *'number-theoretic relation'* R holds (is decidable), there exists a formal proof for the corresponding formal system formula FRM^P.

7.4 Assertion I of the proof of Proposition V

From Assertion I above we give here this assertion of Proposition V as follows:

(7.4.1) $\forall FRM^P, \forall W^F, \forall X,$

(7.4.2) $\exists PRF^P, [PRF^P \, Proof^P \, Subst(FRM^P, W^F, X)]$

(7.4.3) $\quad \Rightarrow \exists \Phi(PRF^P), \{\Phi(PRF^P) \, B \, Sb[\Phi(FRM^P), \Psi(W^F), \Phi(X)]\}$

(7.4.4) $\quad \Rightarrow \exists Y, \{Y \, B \, Sb[T, W^R, \Phi(X)]\}$

(7.4.5) $\quad \Rightarrow Bew\{Sb[T, W\,R, \Phi(X)]\}$

where

- $a \, Proof^P \, b$ means a is a combination of symbols of the formal system **P** that is a complete proof scheme of the formal system **P** for the formula b, and

- PRF^P and FRM^P are variables of the language **PV** - the domain of these variables is symbol combinations of the formal system **P**,

- W^F is a variable of the language **PV**, the domain of which is symbols that are variables of the formal system **P**,

- X, Y and T are all variables of the language **PV** of natural numbers, with the domain of expressions of the language **PV** for natural numbers, and

- by the mapping functions Φ and Ψ, we have
  $$\Phi(FRM^P) = T, \quad \Psi(W^F) = W^R, \quad \Phi(PRF^P) = Y$$

- in step (7.4.4), $Y \, B \, A$ is Gödel's relation 45, 'Y is a proof of the formula X'.
  The step from (7.4.4) to (7.4.5) follows from the definition given by Gödel's relation 46, which is
  $$Bew(A) \equiv \exists Y, (Y \, B \, A).$$

For the specific value v^F of the variable W^F, the assertion is that there is a specific corresponding value 17 under the function Ψ, and this gives:

(7.4.6) $\forall FRM^P, \forall X,$

(7.4.7) $\exists PRF^P [PRF^P \, Proof^P \, Subst(FRM^P, v^F, X)]$

(7.4.8) $\quad \Rightarrow \exists \Phi(PRF^P) \{\Phi(PRF^P) \, B \, Sb[\Phi(FRM^P), \Psi(v^F), \Phi(X)]\}$

(7.4.9) $\quad \Rightarrow \exists Y \{Y \, B \, Sb[T, 17, \Phi(X)]\}$

(7.4.10) $\quad \Rightarrow Bew\{Sb[T, 17, \Phi(X)]\}$

where $17 = \Psi(v^F)$.

v^F is not a variable of the proof language, since it is not subject to a variable quantifier, and neither is it a free variable in the expression (otherwise the expression would not be a proposition - we note that all the expressions 7.4.6 - 7.4.10 are expressions of the language **PV**).

(7.4.11) The function $\Phi(X)$ will give a value that is a natural number in the language **PV** for any given X. A further assertion is that, since the values of X that occur in the function $\Phi(X)$ in the above expression are only symbol combinations for natural numbers, that is, of the form 0, $f0$, $ff0$, $fff0$, $\ldots$, the function $Z(X)$ (Gödel's relation 17) gives the same value as the function $\Phi(X)$ for all such values of the variable X.

Applying this assertion $\Phi(X) \equiv Z(X)$ to the above gives for a specific variable v^F of the formal system **P**:

(7.4.12) $\forall FRM, \forall X,$

(7.4.13) $\exists PRF^P[PRF^P \, Proof^P \, Subst(FRM \, F, v^F, X)]$

(7.4.14) $\quad \Rightarrow \exists \Phi(PRF^P)\{\Phi(PRF^P) \; B \; Sb[\Phi(FRM^P), \Psi(v^F), \Phi(X)]\}$

(7.4.15) $\quad \Rightarrow \exists \Phi(PRF^P)\{\Phi(PRF^P) \; B \; Sb[\Phi(FRM^P), \Psi(v^F), Z(X)]\}$

(7.4.16) $\quad \Rightarrow \exists Y\{Y \; B \; Sb[T, 17, Z(X)]\}$

(7.4.17) $\quad \Rightarrow Bew\{Sb[T, 17, Z(X)]\}$

where the expressions (7.4.12) – (7.4.17) are asserted to be expressions of the language **PV**.

We note at this point that within this section, the meta-language may include *'number-theoretic relations'* as expressions of the meta-language, since this section does not specifically refer to *'number-theoretic relations'* as specific values. We also note that whereas $Bew\{Sb[T, 17, \Phi(X)]\}$ in (7.4.10) is not a *'number-theoretic relation'* since the function $\Phi(X)$ is not a purely *'number-theoretic function'*, the $Bew\{Sb[T, 17, Z(X)]\}$ in (7.4.17) satisfies the definition of a *'number-theoretic relation'*, since the domain for all of the variables of the expression is the domain of natural numbers.

7.5 Assertion II and Assertion III of the Proof of Proposition V

Assertion II of the proof of Proposition V asserts that for every recursive *'number-theoretic relation'*, there is a corresponding formal system formula. This applies, therefore, to recursive *'number-theoretic relations'* with one free variable. Assertion III asserts that if that recursive *'number-theoretic relation'* R holds for some value of that free variable, then there is a formal proof for the corresponding formal system formula FRM^P when the free variable of that formula is substituted by that same value. This results in the proposition:

(7.5.1) $\forall R, \forall X, \exists FRM^P,$

(7.5.2) $\qquad R(X) \Rightarrow \exists PRF^P[PRF^P Proof^P Subst(FRM^P, v^F, X)]$

While it might be asserted that a correspondence exists between R and FRM^P, there is no explanation for how the part of the expression that is $\exists PRF^P[\ldots]$ is generated, or how it follows either from the expression $R(X)$. For a logical analysis, this is unacceptable. However, it is instructive to see how the assertion of Gödel's Proposition V arises, since Gödel's footnote 39 states that *'Proposition V naturally is based on the fact that for any recursive relation R, it is decidable, for every n-tuple of numbers, from the axioms of the [formal system], whether the relation R holds or not'*.[14]

This is a rather roundabout way of asserting that there exists a proof either of the relation R or its negation for any specific values of its free variables, which would give:

(7.5.3) $\forall R, \forall X, \exists FRM^P$

(7.5.4) $\exists PRF^R[PRF^R Proof^R(R(X))] \Rightarrow \exists PRF^P[PRF^P Proof^P Subst(FRM^P, v^F, X)]$

where $a\ Proof^R\ b$ means that a is a combination of symbols of the language of the relation R that is a complete proof scheme of the combination of symbols b of the language of the relation R, and PRF^R is a variable of the language **PV**, the domain of which is symbol combinations of the language of the relation R.

Now, while we now have an explanation for the assertion that there is a valid generation of $\exists PRF^P[PRF^P \ldots]$, there is clearly an implied assertion of a mapping function such that $PRF^P = T^{R-F}[PRF^R]$ and this has not been explicitly included in the above expression. Besides that omission, there is however still no logical explanation for how the expression $Subst(FRM^P, v^F, X)$ arises from the expression $R(X)$. While the symbol X in the expression $R(X)$ can map to the symbol X in the expression $Subst(FRM^P, v^F, X)$, there remains no definition of how a mapping from R to the entities FRM^P and v^F is to be achieved.

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[14] See also Appendix 1: Provability and Truth.

7.6 Clarification of 'Higher-Order Logic'

Proposition V is a proposition with two explicitly stated quantified variables (that is, governed by a universal quantifier '*For all*'), a variable $R(x)$ whose domain is a set of *'number-theoretic relations'* and a variable x whose domain is the set of natural numbers. Such expressions are commonly referred to as being expressions of higher-order or second-order logic, where as well as normal variables, there are relation variables. However, this is simply an appellation which of itself does not confer on the expression any logical validity. Gödel's Proposition V is in a commonly used format, and is of the form:

(7.6.1) $\forall R(x), \forall x, Expression(R, x)$

    where

        $R(x)$ is a variable whose domain is those *'number-theoretic relations'* that have x as a free variable, and

        x is a variable whose domain is the set of natural numbers.

As it stands, such an expression is ambiguous. On the one hand, the variables of the language in which the relation R is stated (of which x is one) are treated as symbols that are not variables of the language of the expression - it is implied by the expression (7.6.1) that: 'R is an expression that according to the rules of the language in which the relation R is expressed contains only one free variable, and that free variable is the symbol x'. Clearly, x cannot be a variable where it occurs in '$\forall R(x)$', since:

(7.6.2) x cannot be a free variable where it occurs in '$\forall R(x)$' , since the entire

           expression (7.6.1) is asserted to be a proposition, and

(7.6.3) x cannot be a bound variable where it occurs in '$\forall R(x)$', since it is not subject

           to a quantifier within the expression '$\forall R(x)$'.

On the other hand, the expression $Expression(R, x)$ appears to be an expression of the language in which the entire expression (7.6.1) is stated, where the symbol x is a variable of the language in which the entire expression is stated. This means that in the entire expression, the symbol x appears to be both a variable and not a variable of the expression at the same time.

Hence the expression as it stands is not a clear proposition, since the determination of which expressions are to be implied by this expression depends on the interpretation of the expression. Similarly, the determination of which expressions are to be expressions that prove the proposition depends on the interpretation of those expressions and the proposition itself. This is particularly so in the determination of whether a symbol is to be perceived as a variable of such expressions. This scenario is clearly not acceptable for logical analysis.

In any case, the above expression (7.6.1) violates a fundamental principle of logical analysis. That fundamental principle is that all occurrences of a symbol that is a variable in an expression may be simply replaced by any other symbol for a variable, provided that that symbol is not already in the

expression. If we try to apply that principle to the naïve expression $\forall R(x), \forall x, Expression(R, x)$, replacing every occurrence of the symbol x where it is a variable by the symbol y this gives:

(7.6.4) $\forall R(x), \forall y, Expression(R, y)$

since in the ambiguous language used, x is not a variable where it occurs in '$\forall R(x)$' whereas the desired result is:

(7.6.5) $\forall R(y), \forall y, Expression(R, y)$

since the implied assertion in all such expressions is that the assertion of the expression also applies to all relations of one free variable, and not only relation expressions with the actual variable symbol used (x in this example).

Various attempts to circumvent this problem have been attempted where the problem is conveniently ignored, with the result that the fundamental properties of propositions and variables are obscured. There is no obvious reason why such misguided attempts have been made, since an expression that unambiguously expresses the desired concept without defining the domain of the variable R in terms of an entity that is elsewhere defined as a variable is readily achieved. The required expressions can readily be made while at the same time retaining quantifiers on the variables of the expression. Instead of asserting that the domain of the relation R is all relations with one free variable x, all that needs to be asserted is that the domain of R is all relations, with an included condition on the entity x, such as

> *'If R is an expression that according to the rules of the language in which the relation R is expressed contains only one free variable, and that free variable is the symbol x, then … '.*

This gives, instead of the expression (7.6.1):

(7.6.6) $\forall R, \{C(R, x) \Rightarrow Expression(R, x)\}$

> where
>
>> R is a variable whose domain is those *'number-theoretic relations'* , and
>> $C(R, x)$ is the condition that x is a free variable in the relation R, and there are no other free variables in R.

We note furthermore that it is not necessary to stipulate that the variable R in the above expressions has the domain of *'number-theoretic relations'*, since this can be included in the condition, viz:

(7.6.7) $\forall Y, \{C(Y, x) \Rightarrow Expression(Y, x)\}$

> where $C(Y, x)$ is the condition that Y is a *'number-theoretic relation'*, x is a free variable in the relation Y, and there are no other free variables in Y.

In the following for convenience, we refer to the R in the expressions as having the domain of *'number-theoretic relations'*, though in principle, the requirement that R be a *'number-theoretic relation'* could be included in the condition C.

In principle we assume that there is no difficulty in precisely defining a condition such as $C(R, x)$; in any case, such a definition is required to define the domain of R in Gödel's naïve expression (7.6.1). When the expression is stated in this format, it is quite evident that x cannot be a variable anywhere in the expression since:

(7.6.8) x cannot be a free variable in $C(R, x)$, since the expression is a proposition, and

(7.6.9) x cannot be a bound variable in $C(R, x)$, since it is not subject to a quantifier.

It follows that in $Expression(R, x)$ any reference to x is to x as an entity that is not a variable of the entire expression, since the expression is now expressed in a clearly defined fashion.

The expression (7.6.6) implies the expression given when any valid specific value of R is substituted for R, for example:

(7.6.10) $C(R_1, x) \Rightarrow Expression(R_1, x)$

   where R_1 is any valid value of the variable R.

Furthermore, there may be expressions with other quantified variables (that are not variables in the relation R), for example,

(7.6.11) $\forall R, \forall Y, \{C(R, x) \Rightarrow Expression(R, Y, x)\}$

There is also an implied assertion in all expressions such as $\forall R(x), \forall x, Expression(R, x)$ that the expression where the symbol x is replaced by some other variable symbol[15] is also a valid expression. We may readily apply this principle to the expression (7.6.6), which is $\forall R, \{C(R, x) \Rightarrow Expression(R, x)\}$.

Since x is not a variable in the expression, we may generalize the expression, by the introduction of another quantified variable, which is readily assimilated into the expression (7.6.6), giving:

(7.6.12) $\forall R, \forall W, \{C(R, W) \Rightarrow Expression(R, W)\}$

   where

      R is a variable whose domain is *'number-theoretic relations'*,

      W is a variable whose domain is symbols for variables in *'number-theoretic relations'*, and

      $C(R, W)$ is the condition that W is a free variable in the relation R.

The above deals with the case where an expression explicitly references the symbol of a variable such as x in the relation subject to the quantifier, as in $\forall R(x)$. However, simply removing that explicit reference to a symbol such as x does not ensure an absence of ambiguity in the expression.

As a case in point, we may consider an example of a higher-order formula - an expression referred to as the axiom of induction, which is commonly stated as:

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[15] Where the symbol is a variable in the same language as the language in which x is a variable symbol.

(7.6.13) $\forall R, \{(R(0) \wedge \forall x[R(x) \Rightarrow R(x+1)]) \Rightarrow \forall x R(x)\}$

which is interpreted as: *'For every relation R, if R is true for 0, and if for every x, $R(x)$ implies $R(x+1)$, then R is true of every x.'*

However, there is an implicit assertion that R, where referenced by '$\forall R$', is a relation with only one free variable, but this is not explicit in the expression as given. If we express this explicitly by applying a condition on R as *'If R is a relation with one free variable, then ...'* we get:

(7.6.14) $\forall R, \{C(R) \Rightarrow [(R(0) \wedge \forall x[R(x) \Rightarrow R(x+1)]) \Rightarrow \forall x R(x)]\}$

where $C(R)$ is the condition that there is one free variable in the relation R.

That condition will express, *'There exists a W, such that W is a free variable in R, and there does not exist a Z, such that $Z \neq W$, and Z is a free variable in R'* , which gives:

(7.6.15) $C(R) \equiv \exists W, Fr(R, W) \wedge \neg \exists Z, [(Z \neq W) \wedge Fr(R, Z)]$

where $Fr(R, W)$ means that W is a free variable in R.

Again, since the condition is implicit in the original expression (7.6.13), if the original expression is a valid logical expression, that condition must in principle be expressible.

Clearly, if the entire expression is to be a valid expression, this condition on R must be explicitly expressible as part of the expression, since the condition on R is implicit in the expression. When stated explicitly, it is evident that there must be some variable such as W in order to express the condition. The domain of this variable W is symbols that may be variables in the relation R. This also means that if we have an expression such as $\forall W, Expression(W)$ which is a valid proposition (and hence has no free variable) in the same language as that of the expression (7.6.14), that expression implies that the statement $Expression(c)$, where c is some member of the domain of W, is a proposition with no free variables. It follows that c cannot be a variable in the language of the expression (7.6.14), since it is neither subject to a quantifier nor is it a free variable (otherwise the expression $Expression(c)$ would not be a proposition).

This means that in the expression:

(7.6.16) $\forall R, \{C(R) \Rightarrow [(R(0) \wedge \forall x[R(x) \Rightarrow R(x+1)]) \Rightarrow \forall x R(x)]\}$

which is equivalent to

(7.6.17) $\forall R, \{(\exists W, Fr(R, W) \wedge \neg \exists Z, [(Z \neq W) \wedge Fr(R, Z)]) \Rightarrow$
$$[(R(0) \wedge \forall x[R(x) \Rightarrow R(x+1)]) \Rightarrow \forall x R(x)]\}$$

that either:

(7.6.18)   the language of the entire expression is unambiguous and the symbol x cannot be a variable in the language of the entire expression, since it is a member of the domain of the variable W, or

(7.6.19)   the entire expression is in an ambiguous language. However, the rules for this ambiguous language are not clearly defined, and the notion that this is a suitable basis for a language for logical analysis is absurd. Nor has been shown that such a language is in any way necessary for logical analysis - that is, it has not been shown that there is some deficiency in a language that is clearly defined.[16]

To state the expression in a language that is unambiguous, if it is acknowledged that the language of the relation R is a sub-language (which we call the language S) to the language of the entire expression, this gives:

(7.6.20) $\forall R, \forall W,$

$$\{C(R, W) \Rightarrow Provable^S[R(0)] \wedge Provable^S[\forall^S W(R(W) \Rightarrow^S R(W + 1))]$$
$$\Rightarrow IC^S[\forall^S W R(W)]\}$$

where

- $\forall^S$ is the symbol for 'For all' in the language S,
- $\Rightarrow^S$ is the symbol for 'implies' in the language S,
- W is a variable whose domain is symbols for variables of the language S,
- $C(R, W)$ is the condition that W is a free variable in the relation R, and there are no other free variables in R,
- $Provable^S(a)$ means that a is an expression that is provable according to the axioms and rules of the language S as already defined.
- we use the terminology $R(0)$ to represent $Subst(R, W, 0)$,
  $R(W + 1)$ to represent $Subst(R, W, W + 1)$, and
  $R(W)$ to represent R.

These are all expressions in the meta-language, which represent expressions of the sub-language S upon substitution of the meta-language variable W by a symbol that is a variable in the sub-language S.

We note that the above expression may be considered to be a rule of inference, rather than an axiom, so that $IC^S(a)$ means that a is an formula that is an immediate consequence of the other expressions $R(0)$ and $\forall^S W(R(W) \Rightarrow^S R(W + 1))$ of the language S, where W is some variable symbol of the language S. In general, if a proof exists for a formula, then there will be a series of formulas, each of which is either an axiom or is an immediate consequence of previous formulas. Alternatively, we could state that $IC^S\{a, R(0), \forall^S W(R(W) \Rightarrow^S R(W + 1))\}$ means that a is an immediate consequence of the expressions $R(0)$ and $\forall^S W(R(W) \Rightarrow^S R(W + 1))$, provided $R(0)$ and $\forall^S W(R(W) \Rightarrow^S R(W + 1))$ are either themselves immediate consequences or axioms.

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[16] Of course, in one sense, it could be said that this is what Gödel's proof asserts. However, an assertion by a vaguely defined ambiguous language that that language itself is required to 'prove' that that language itself is somehow superior to a language that is not ambiguous is quite clearly absurd.

Again, we use different symbols for relational operators to avoid having symbols whose syntactical interactions are ambiguous. For a given variable symbol x of the language $\mathbf{S}$, the expression (7.6.20) implies:

$$(7.6.21) \quad \forall R, \{C(R, x) \Rightarrow Provable^S[R(0)] \land Provable^S[\forall^S x(R(x) \Rightarrow^S R(x+1))]$$
$$\Rightarrow IC^S[\forall^S x R(x)]\}$$

It will be noted that the expression (7.6.21) is very similar to the expression (7.6.13) above, but without that expression's ambiguity; this demonstrates how logically coherent language can give rise to logically acceptable expressions.

7.7 'Higher Order Logic' and Assertion III of Proposition V

Gödel's Proposition V is a proposition with two explicitly stated quantified variables, R and X. We have already seen in Section 7.5 that the implication is that R is referenced by the expression $\exists PRF^R[PRF^R Proof^R(R(X))]$. There is no difficulty in expressing the proposition with a condition on the symbol X, as in Section 7.6, rather than the implied assertion that the domain of R is the domain of all relations with the free variable X. The assertion of Proposition V as given by Gödel in naïve format is:

(7.7.1) $\forall R(x), \forall x, \exists u, \exists r,$

(7.7.2) $\qquad R(x) \Rightarrow Bew\{Sb[r, u, Z(x)]\}$

In this naïve expression of Gödel's proof, when an expression such as $y\,B\,\{Sb[m, n, Z(x)]\}$ is defined and it is asserted that it is a *'number-theoretic relation'* and which is a relation referred to by $R(x)$ in Proposition V, that is on the basis that:

> '$\forall R(x), Expression[R(x)]$' implies the expression obtained when the free variable $R(x)$ of **Expression** has been substituted by a permissible specific value for that variable.

That means that when Gödel asserts that a value such as $y\,B\,\{Sb[m, n, Z(x)]\}$ is a value that may be referred to by $R(x)$ in his Proposition V, that means that it may be substituted for any instance of the variable $R(x)$ in **Expression**. It also means that in that new **Expression**, that value of $R(x)$ cannot change, since it is a specific value. However, according to Gödel, x is still a variable in the expression $y\,B\,\{Sb[m, n, Z(x)]\}$.

But if that is the case, and it is also the case that $y\,B\,\{Sb[m, n, Z(x)]\}$ is a specific value, that implies that it is immaterial what value is subsequently given for x (since specific values always remain the same specific values within the meta-language). That would mean that when we substitute another value for x, such as q, it can be said that $y\,B\,\{Sb[m, n, Z(q)]\}$ is the same specific value as $y\,B\,\{Sb[m, n, Z(x)]\}$, and that $y\,B\,\{Sb[m, n, Z(x)]\}$ is the same specific value for every value of x. But that is clearly absurd.

The expression of (7.7.1) – (7.7.2) can be expressed in a more logically coherent manner to give:
(7.7.3) $\forall R, \forall X, \exists u, \exists r$

(7.7.4) $\quad C(R, x) \Rightarrow \exists PRF^R\{PRF^R Proof^R[Subst(R, x, X)]\} \Rightarrow Bew\{Sb[r, u, Z(X)]\}$
> where $C(R, x)$ is the condition that x is a free variable in the relation R, and there are no other free variables in R. Note that in the above expression, the assertion that there exists a mapping function that maps the *'number-theoretic relations'* to the entities r, u and X is not explicitly expressed.

We can also assert that the above applies for every relation R with one free variable, rather than only relations with the free variable x, an assertion that is implied by Gödel's naïve expression of Proposition V:

(7.7.5) $\forall R, \forall W, \forall X, \exists u, \exists r,$

(7.7.6) $C(R, W) \Rightarrow \exists PRF^R\{PRF^R Proof^R[Subst(R, W, X)]\} \Rightarrow Bew\{Sb[r, u, Z(X)]\}$

    where W is a variable whose domain is symbols for variables that are defined for relations R.

We now have a logically coherent derivation of an expression regarding formal system expressions, from an expression regarding *'number-theoretic relations'*, which is:

(7.7.7) $\forall R, \forall X, \exists u, \exists r,$

(7.7.8) $\qquad C(R, x) \Rightarrow \exists PRF^R\{PRF^R Proof^R[Subst(R, x, X)]\}$

(7.7.9) $\qquad\qquad \Rightarrow \exists PRF^P[PRF^P Proof^P Subst(FRM^P, v^F, X)]$

And combining the above with those from Section 7.4 above, we have:

(7.7.10) $\forall R, \forall X, \exists u, \exists r,$

(7.7.11) $\qquad C(R, x) \Rightarrow \exists PRF^R\{PRF^R Proof^R[Subst(R, x, X)]\}$

(7.7.12) $\qquad\qquad \Rightarrow \exists PRF^P[PRF^P Proof^P Subst(FRM^P, v^F, X)]$

(7.7.13) $\qquad\qquad \Rightarrow \exists \Phi(PRF^P)\{\Phi(PRF^P) \ B \ Sb[\Phi(FRM^P), \Psi(v^F), \Phi(X)]\}$

(7.7.14) $\qquad\qquad \Rightarrow \exists \Phi(PRF^P)\{\Phi(PRF^P) \ B \ Sb[\Phi(FRM^P), \Psi(v^F), Z(X)]\}$

(7.7.15) $\qquad\qquad \Rightarrow \exists Y\{Y B Sb[T, u, Z(X)]\}$

(7.7.16) $\qquad\qquad \Rightarrow Bew\{Sb[T, u, Z(X)]\}$

However, as already noted, the quantifier on R means that *'number-theoretic relations'* are specific values of the meta-language. That means that while the expressions (7.7.10) – (7.7.12) are propositional expressions of the meta-language **PV**, the expressions (7.7.15) – (7.7.16) satisfy the definition of *'number-theoretic relations'*, since their variables all are defined in the field of natural numbers. The inherent flaw in Gödel's Proposition V becomes apparent in several different ways. Section 6 demonstrated one such anomaly. Further anomalies can be shown as in the following sections.

7.8   The assumption of equivalence of the Φ Function and the Z function

It has already been demonstrated in section 5A that Gödel's assumption of equivalence of his Z function and his numbering function Φ is not only without foundation, but is completely contrary to any logical implication, as it conflates different levels of language. This section provides further demonstrations of the impossibility of such equivalence.

As shown in (7.4.11) above, Proposition V relies, among other things, on a claim of equivalence of the Gödel numbering function and the Z function (Gödel's relation 17). That is, it must be asserted in that for any value of X that is a number, $\Phi(X) = Z(X)$. That expression cannot be a *'number-theoretic'* expression, since the Gödel numbering function $\Phi(X)$ refers to formal language symbols that are not numbers. So the assertion that $\Phi(X) = Z(X)$ must be an assertion of the meta-language **PV** of Proposition V, so that the meta-language **PV** makes the assertion:

(7.8.1) *For every X, where X is a number, $\Phi(X) = Z(X)$*

Consider the Gödel Numbering function, $\Phi(X)$; the domain of the free variable X is symbols of the formal language and combinations of symbols of the formal language. That means that $\Phi(X)$ is an expression of the meta-language; it cannot be an expression in the language of *'number-theoretic relations'* and not an expression in the formal language. And that means that the variable X in $\Phi(X)$ is a variable of the meta-language.

Now consider the function $Z(X)$. This function is a *'number-theoretic function'*, listed as Gödel's *'number-theoretic relation'* 17. That means that it is not an expression of the meta-language **PV**, but an expression of the sub-language of *'number-theoretic relations'*. That means that the variable X in $Z(X)$ is a variable of a *'number-theoretic relation'*.

This means that when it is asserted as in (7.8.1) that:

For every X, where X is a number, $\Phi(X) = Z(X)$,

it follows that in the Gödel Numbering function $\Phi(X)$, which is an expression of the meta-language, X has to be a variable of the meta-language. But it also follows that in the function $Z(X)$, X has to be a variable of a *'number-theoretic relation'* - it has to be a variable of a sub-language.

That means that the expression (7.8.1) *For every X, where X is a number, $\Phi(X) = Z(X)$* cannot be a valid proposition. This can be further shown when we ensure that we use symbols for the variables of the meta-language that are different to the symbols for variables of the language of *'number-theoretic relations'*. When this is the case, the expression (7.8.1) above cannot even be expressed; for suppose that X is a variable of the meta-language - that means that the variable in the *'number-theoretic relation'* Z cannot be X. This means that Gödel's proof of Proposition V is incorrect, since it relies on the expression (7.8.1).

The problem that arises from the assumption of equivalence of the Φ function and the Z function can be demonstrated in other ways:

As stated above, Gödel's Proposition V requires that $Z(X)$ is a *'number-theoretic relation'*. It might be thought once the variable X is substituted by some number value, say 5234, that the resultant expression can then be a *'number-theoretic relation'*. But besides the variable X, the definition of $Z(X)$ includes other variables as bound variables. And since the assertion that $\Phi(X) = Z(X)$ is made in the meta-language **PV**, it follows that all the bound variables in the definition of $Z(X)$ are also variables of the language **PV**. And from that it follows that the simple substitution of the free variable of $Z(X)$ results in an expression, such as $Z(5234)$ that is still an expression of the language **PV** and hence it cannot be at the same time an expression of a sub-language, and hence it cannot be a *'number-theoretic relation'*. From this it follows that Gödel's Proposition V entails an assumption that under rigorous logical analysis cannot be upheld.

We may also see this is another way, as follows:
Besides the variable X, the definition of $\Phi(X)$ includes other variables as bound variables, variables that may represent values other than numbers. That means that even if we take for the value of X a specific number, the Gödel numbering function is still not a *'number-theoretic relation'*. For example, $\Phi(5234)$, $\Phi(905781)$ and $\Phi(135874)$ are not *'number-theoretic relations'*. But when Gödel claims that $\Phi(5234)$ is equivalent to $Z(5234)$, and if $Z(5234)$ can be referred to by the meta-language as a *'number-theoretic relation'*, then $\Phi(5234)$ must also be a *'number-theoretic relation'*. Otherwise they cannot be equivalent in the meta-language **PV** of Proposition V. But $\Phi(5234)$ cannot be a *'number-theoretic relation'*. That is a contradiction.

Alternatively, we may consider the claim of equivalence of the Gödel numbering function and the Z function as follows:
The expression $Bew\{Sb[T, 17, Z(X)]\}$ is a *'number-theoretic relation'*, since its variables all have the domain of natural numbers, and where X and T are variables with the domain of natural numbers.

At the same time, it is asserted as in (7.4.14) – (7.4.17) that:

(7.8.2) $Bew\{Sb[\Phi(FRM^P), \Psi(v^F), \Phi(X)]\} \Rightarrow Bew\{Sb[T, 17, Z(X)]\}$

where it is asserted that T is the number given by the Gödel mapping function Φ on some formal system formula FRM^P, that is that $T = \Phi(FRM^P)$, and that 17 is the number given by the Gödel mapping function Ψ, that is that $17 = \Psi(v^F)$. But if it is asserted that:

(7.8.3) $Bew\{Sb[\Phi(FRM^P), \Psi(v^F), \Phi(X)]\} \Rightarrow Bew\{Sb[T, 17, Z(X)]\}$

and also that

(7.8.4) $Bew\{Sb[T, 17, Z(X)]\} \Rightarrow Bew\{Sb[\Phi(FRM^P), \Psi(v^F), \Phi(X)]\}$

then equivalence is being asserted, i.e., it is being asserted that

(7.8.5) $Bew\{Sb[T, 17, Z(X)]\} \equiv Bew\{Sb[\Phi(FRM^P), \Psi(v^F), \Phi(X)]\}$

And if equivalence of the two expressions is being asserted then the expression

$$Bew\{Sb[\Phi(FRM^P), \Psi(v^F), \Phi(X)]\}$$

must also satisfy the definition of a *'number-theoretic relation'*, even though it includes variables that do not have the domain only of natural numbers (or, if the variables are substituted, includes values that are not numbers).

Now, $Bew\{Sb[\Phi(FRM^P), \Psi(v^F), \Phi(X)]\} \Rightarrow Bew\{Sb[T, 17, Z(x)]\}$ must be asserted as in (7.4.14) – (7.4.17).[17] That means that to avoid a contradiction in the definition of a *'number-theoretic relation'* , it would have to be asserted that:

$$Bew\{Sb[T, 17, Z(X)]\} \quad \text{does not imply} \quad Bew\{Sb[\Phi(FRMF), \Psi(v^F), \Phi(X)]\}.$$

But that of course, would mean that we cannot assert that $T = \Phi(FRM^P)$ and $17 = \Psi(v^F)$; that is, that we cannot assert that the Gödel numbering system is a mapping function (or the Ψ function) at the same time as defining $Bew\{Sb[T, 17, Z(X)]\}$ as a *'number-theoretic relation'*.

7.9 The impossibility of Gödel's result

It might be thought from a cursory examination of Gödel's Propositions V and VI that the meta-linguistic conflation is immaterial since the results of the numerical values of the Φ and the Z functions appear to be identical. But if the format of the domain of the variables are not identical, then instead of claiming that the function Z is a direct replacement of the function Φ (as Gödel does) one has to claim that:

$$Z(C(X)) = \Phi(X)$$

where $C(X)$ converts an expression in the format of the formal system **P** to the format of the system of *'number-theoretic relations'*, and where X has the domain of numbers in the format of the formal system **P**.

But we can then see that even if we make the assumption that an expression including $Z(C(X))$ is a *'number-theoretic'* expression, Gödel's result cannot ensue. This is because in his Proposition VI, he uses the expression:

$$x \, B_c \, Sb(y, 19, Z(y))$$

where he relies on this expression having only *two* free variables. But the requirement of a conversion function such as $C(X)$ means that the expression should be:

$$x \, B_c \, Sb(y, 19, Z[C(Y)])$$

which has *three* free variables, and the variables y and Y cannot be identical. But since Proposition VI requires that this expression must have only two variables, the result of Proposition VI cannot be obtained.[18] It can be seen that Gödel's assumption that the second and third variables can be identical is another erroneous conflation of levels of language that follows directly from the erroneous assumption of equivalence of the Z and Φ functions.

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[17] Again, due to the brevity of the outline that Gödel gives for Proposition V, this is not explicitly stated in Gödel's proof, but it is necessarily implied by the notion of a mapping from the formal system **P** by means of the Gödel numbering function.

[18] See equations numbers 3 and 4 in Gödel's Proposition V and the corresponding equations 9 and 10, and 15 and 16 in Proposition VI.

7.10 Assertions of truth rather than provability

Gödel's proof refers to *'number-theoretic relations'* holding or their negation holding (that the relations are 'true' or 'false'), while formal formulas are referred to as being provable or not provable. One might expect that if formulas of the formal language can be expressions regarding numbers, that such expressions would be *'number-theoretic relations'* (see also Appendix 2: The Formal System and 'Number-Theoretic Relations'). But even if by some definition formal formulas can be excluded from the definition of *'number-theoretic relations'*, there is still no reason as to why one should refer to expressions of one sub-language (formal formulas) as being provable or not provable, and those of another sub-language (*'number-theoretic relations'*) as being true or false (see also Appendix 1: Provability and Truth).

In the language **PV** of Gödel's Proposition V formal formulas and *'number-theoretic relations'* are all simply combinations of symbols, which are specific values. For example, Gödel refers to expressions such as $\neg Bew\{Sb[x, 17, Z(y)]\}$, asserting that it is a *'number-theoretic relation'* which has a corresponding Gödel number. He also asserts that it must be either true or false. But that is a contradiction. If it is a *'number-theoretic relation'* then it is seen by the meta-language as a specific value. It is not an expression that has a syntactical relationship to any other expression of the meta-language other than as a specific value; it is simply a combination of symbols to which the meta-language cannot assign a determination of true nor false other than by a determination of the existence of a set of *'number-theoretic relations'* for which an appellation of 'proof' might be applied, which would allow the term 'provable' to be applied to the combination of symbols in question.

On the other hand, if the expression is not a *'number-theoretic relation'* then Proposition V cannot give a corresponding Gödel number, for the assertion is that for every *'number-theoretic relation'* there is a matching Gödel number.

7.11 Confusion regarding 'number-theoretic relations'

Consider any number that is a Gödel number or a number given by the function Ψ. Since for any such number x in the meta-language **PV**, either $x = \Psi(\Omega)$, or $x = \Phi(\Omega)$ where Ω is some symbol or combination of symbols of the formal system **P**.

That means that in the meta-language, any expression that includes such a number x is equivalent to the expression obtained when that number x is substituted by $\Psi(\Omega)$ or $\Phi(\Omega)$ as appropriate. And that means that if those expressions can be termed *'number-theoretic relations'* this results in a contradiction, since in the meta-language **PV** they are equivalent to expressions that refer to entities other than numbers.

Similarly, if a *'number-theoretic relation'* R is defined as $\neg Bew\{Sb[q, 17, Z(q)]\}$, and if it is defined that for some formal formula FRM there exists a number q such that $q = \Phi[FRM]$, and if there is a function λ that gives a formal formula FRM that corresponds to a *'number-theoretic relation'* R, then we have $FRM = \lambda(R)$, so that we have:

(7.11.1) $R = \neg Bew\{Sb[q, 17, Z(q)]\}$

and

(7.11.2) $FRM = \lambda(R) = \lambda(\neg Bew\{Sb[q, 17, Z(q)]\})$

and

(7.11.3) $q = \Phi[FRM] = \Phi[\lambda(\neg Bew\{Sb[q, 17, Z(q)]\})]$

so that, from (7.11.1), on substituting $\Phi[FRM]$ for q, and $\lambda(R)$ for FRM, and $\neg Bew\{Sb[q, 17, Z(q)]\}$ for R, we have:

(7.11.4)

$R = \neg Bew\{Sb[\Phi[\lambda(\neg Bew\{Sb[q, 17, Z(q)]\})], 17, Z(\Phi[\lambda(\neg Bew\{Sb[q, 17, Z(q)]\})])]\}$

and

(7.11.5) $R = \neg Bew\{Sb[\Phi[\lambda(R)], 17, Z(\Phi[\lambda(R)])]\}$

where R is defined in terms of itself, and where R is supposedly a *'number-theoretic relation'*.

But since R is equivalent to the above expression (in the language of the expression, which is the language **PV**), that expression must also be a *'number-theoretic relation'*, which of course is a contradiction, since it refers to entities other than numbers.

Appendix 1:   Provability and Truth

When Gödel states in his footnote 39 that *'Proposition V naturally is based on the fact that for any recursive relation R, it is decidable ... from the axioms of the [formal system], whether the relation R holds or not.'*, it appears at first glance that Gödel is relying on the relation R being *'decidable from the axioms of the [formal system]'* rather than on the notion of truth.

But while Gödel avoids the use of the term *'true'* in relation to this proposition, the term is nonetheless implied by the Proposition and the following text. In the proposition, it is implied that if $R(x)$ *'holds'* or is *'true'*, then $Bew\{Sb[r, u_n, Z(x)]\}$ *'holds'* or is *'true'*, But there is no clear definition of what is being asserted by the unqualified reference to $R(x)$, or what is being asserted by the implied terms *'holds'*, and *'true'*.

Either the unqualified reference to the term $R(x)$ is equivalent to stating that the relation $R(x)$ is *'provable from the axioms of the [formal system]'* or it is not. Or we might say, either the terms *'holds'* and *'true'* are equivalent to the term *'provable from the axioms of the [formal system]'* or they are not.

According to what we are given by Gödel, it is clear that any 'number-theoretic relation' which is *'provable from the axioms of the [formal system]'* can be said to *'hold'*, or to be *'true'*. This means that if the terms *'hold'*, *'true'* and *'provable from the axioms of the [formal system]'* are not equivalent, then it must be the case that there are 'number-theoretic relations' which *'hold'* and are *'true'*, but are not *'provable from the axioms of the [formal system]'* (and which are not themselves axioms).

But this means that, regardless of whatever definition is actually applied to the terms *holds'* or *true'*, it must be the case that there are 'number-theoretic relations' which *hold'* and are *true'*, but are not *provable from the axioms of the [formal system]'*, and which are not axioms.

But if the initial assumption is that there are 'number-theoretic relations' which *hold'* and are *true'*, but are not *provable from the axioms of the [formal system]'*, and which are not axioms, the rest of Gödel's proof is completely pointless. For if a proof asserts there exists some formula of the formal system **P**, which is not an axiom, which is not *provable from the axioms of the [formal system]'*, but nonetheless *holds'* and is *true'*, and that assertion itself relies on the assumption that there exists at least one 'number-theoretic relation', which is not an axiom, which *holds'* and is *true'*, but is not *provable from the axioms of the [formal system]'*, then it proves nothing at all.

Appendix 2:   The Formal System and 'Number-theoretic Relations'

It must be the case that variables of the formal system **P** cannot be included as variables of *'number-theoretic relations'* in that definition, without contradictions arising such as in the following:

Although Proposition V is expressed in terms of a variable x, there is an implied assertion that the proposition applies for any valid variable of the language in which Proposition V is expressed. Hence if, for example, a variable v_1 of the formal system **P** was defined as a valid variable of the language of *'number-theoretic relations'* defined by the entity R, we would have, given Gödel's expression of Proposition V:

(A.2.1) $\forall R(v_1), \forall v_1, \exists FRM^F, \exists r$

(A.2.2) $R(v_1) \Rightarrow \exists PRF^P[PRF^P Proof^P Subst(FRM^F, v^F, v_1)]$

(A.2.3) $\Rightarrow Bew\{Sb[r, u, Z(v_1)]\}$

> where v_1 is a variable of the formal language, and also a variable of *'number-theoretic relations'* and also a variable of the language **PV** of the proposition.

Proposition V asserts that the variable v^F (and the corresponding u) may be chosen arbitrarily so that v_1 could be chosen as the variable v^F. But if the variable v_1 is a bound variable in the proposition, if v^F is chosen to be v_1, the following expression would be a *'proposition'*:

(A.2.4) $\forall R(v_1), \forall v_1, \exists FRM^F, \exists r,$

(A.2.5) $R(v_1) \Rightarrow \exists PRF^P[PRF^P Proof^P Subst(FRM^F, v_1, v_1)]$

(A.2.6) $\Rightarrow Bew(Sb[r, u_1, Z(v_1)])$

> where the *'variable'* v_1 is substituted by the *'variable'* v_1.

This is clearly an illogical construct.

That means that any definition of *'number-theoretic relations'* must exclude all expressions of the formal system, and since any formal system may be chosen as the subject of Gödel's proof, that means that any definition of *'number-theoretic relations'* must exclude all expressions of any formal system.

Appendix 3: A Simplified Version of Gödel's Proof

A full understanding of Gödel's Proposition V leads to a much simpler proof of Gödel's principal result (as given by his Proposition VI), which can be obtained without any reference to recursion or ω-consistency, but which uses the principles of the outline of Gödel's proof. We follow certain principles as used in Gödel's outline proof for his Proposition V, and for his Proposition VI:

(A.3.1) Given a *'number-theoretic relation'* $R(x)$, with the free variable x, there exists a corresponding formal system formula $F(v)$, with the free variable v.[19] The free variable v may be chosen arbitrarily.

(A.3.2) For the formal system formula $F(v)$, there exists, by the Gödel numbering function, a corresponding Gödel number r, with a prime factor p^u, where u is the number defined to correspond to the variable v.

(A.3.3) Given such a Gödel number r, with a prime factor p^u, where u is the number defined to correspond to the variable v, there exists a corresponding formal system formula, and hence there also exists a corresponding *'number-theoretic relation'* $R(x)$ for any such Gödel number r.[20]

(A.3.4) If the formal formula $F(v)$ that corresponds to a *'number-theoretic relation'* $R(x)$ is provable for a specific value of v, it follows that the *'number-theoretic relation'* $R(x)$ must be true for that value of x, if the formal system is consistent.

(A.3.5) The *'number-theoretic relation'* $Bew\{Sb[r, u, Z(x)]\}$ corresponds to the proposition

> *'There exists a proof in the formal system of the formal system formula B (where r is the Gödel number of that formula) where its free variable (which is the variable of the formal system that corresponds to u) has been substituted by some number x'.*

For any formula of the formal system, either there exists a formal proof sequence of that formula or there does not. It follows that for any number n that is the Gödel number for a formal system formula, either $Bew(n)$ or $\neg Bew(n)$ is true; if the number n is not a Gödel number for a formal system formula, then $\neg Bew(n)$ is true. It follows that for any number n, either $Bew(n)$ or $\neg Bew(n)$ is true.

Since $Sb[w, 17, Z(x)]$ is a *'number-theoretic function'* which has a number value for any w and x, it follows that:

For all w, for all x, either:

(A.3.6) $Bew\{Sb[w, 17, Z(x)]\}$ or

(A.3.7) $\neg Bew\{Sb[w, 17, Z(x)]\}$

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[19] We note that this does not require that the number relation be recursive, only that there exists a corresponding specific formula of whichever formal system is to be the subject of the proof. The relation we subsequently use is the relation $Bew\{Sb[x, 17, Z(x)]\}$, which is a *'number-theoretic relation'* defined in terms of simpler arithmetical functions and relations. Hence the proof applies to any formal system that can formulate such terms.

[20] If there is more than one such *'number-theoretic relation'*, one may always be selected, arbitrarily or for example, by alphabetical/alphanumeric ordering.

Given a specific value of w, if $Bew\{Sb[w, 17, Z(x)]\}$ is true for some specific value of x, then any number-theoretic relation $R(x)$ that corresponds to the specific number w is also true for that value of x, so that we have:

For a specific value r , for all x, either:

(A.3.8) $Bew\{Sb[r, 17, Z(x)]\} \Rightarrow R(x)$ or

(A.3.9) $\neg Bew\{Sb[r, 17, Z(x)]\}$

    where $R(x)$ is a number-theoretic relation that corresponds to the number r.

(A.3.10) We define a relation with one free variable x as $\neg Bew\{Sb[x, 17, Z(x)]\}$, and we define q as a corresponding Gödel number for this relation $\neg Bew\{Sb[x, 17, Z(x)]\}$.[21]

Substituting q for w in A.3.6 - A.3.7 gives:

(A.3.11) $Bew\{Sb[q, 17, Z(x)]\} \Rightarrow R(x)$ or

(A.3.12) $\neg Bew\{Sb[q, 17, Z(x)]\}$

    where $R(x)$ is a number-theoretic relation that corresponds to the number q.

Since $\neg Bew\{Sb[x, 17, Z(x)]\}$ is defined as a number-theoretic relation that corresponds to the number q, it follows that:

For any specific value of x, either:

(A.3.13) $Bew\{Sb[q, 17, Z(x)]\} \Rightarrow \neg Bew\{Sb[x, 17, Z(x)]\}$ or

(A.3.14) $\neg Bew\{Sb[q, 17, Z(x)]\}$

So, for $x = q$ either:

(A.3.15) $Bew\{Sb[q, 17, Z(q)]\} \Rightarrow \neg Bew\{Sb[q, 17, Z(q)]\}$ or

(A.3.16) $\neg Bew\{Sb[q, 17, Z(q)]\}$

It follows that (A.3.15) cannot apply, since that is a straightforward contradiction. Therefore it must be the case that (A.3.16) applies. This means that the relation $\neg Bew\{Sb[q, 17, Z(q)]\}$ is true. Since that is the case, then there cannot be a formal proof of the formal formula that corresponds to $Sb[q, 17, Z(q)]$.

Now the formal formula that corresponds to $Sb[q, 17, Z(q)]$ is the formal formula that corresponds to the relation $R(q)$. But the relation $R(q)$ is the relation $\neg Bew\{Sb[q, 17, Z(q)]\}$, which is true, as given by (A.3.16).

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[21]The number q is selected such that the only free variable v of the corresponding formal formula corresponds by the Ψ function to the number 17.

Therefore the relation $R(q)$ is true, and the formal formula that corresponds to $R(q)$ must also be true, but there cannot be a formal proof of the formal formula that corresponds to $R(q)$.

This completes the proof.[22] It will be observed that the above proof requires no additional assumptions that are not of the same nature as those used in Gödel's original proof.

In this simplified version, we note that the relation $R(q)$ is asserted at the same time to correspond, by the same mapping, to both the expression $Sb[q, 17, Z(q)]$ and the expression $\neg Bew\{Sb[q, 17, Z(q)]\}$. As well as not being equivalent *'number-theoretic relations'*, $Sb[q, 17, Z(q)]$ is a function that represents a specific number value, whereas the expression $\neg Bew\{Sb[q, 17, Z(q)]\}$ is supposedly an expression that does not have a number value. Such anomalies are also present in Gödel's original version, but are not so readily apparent as they are here.

References

[1] Gödel, Kurt: 'On Formally Undecidable Propositions of Principia Mathematica and Related Systems', 1931, English translation by B. Meltzer, Oliver & Boyd, Edinburgh, 1962

[2] Gödel, Kurt: 'On Formally Undecidable Propositions of Principia Mathematica and Related Systems', 1931, English translation by Martin Hirzel, 27 Nov 2000
url: http://www.research.ibm.com/people/h/hirzel/papers/canon00-goedel.pdf

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[22]Note that, unlike Gödel's Proposition V, we cannot assert that the converse of A.3.15, that is, we cannot assert in A.3.16 that
$\neg Bew\{Sb[q, 17, Z(q)] \Rightarrow \neg Bew\{Sb[q, 17, Z(q)]\}$.