On Smith-Volterra-Cantor sets and their measure

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Abstract

This paper analyzes common definitions of the Cantor Middle Thirds set and Smith-Volterra-Cantor sets in general (of which the Cantor Middle Thirds set is a special case) and demonstrates that there are implicit assumptions associated with such definitions which are logically unsustainable.

1 The Cantor Middle Thirds Set

We begin with an analysis of the Cantor Middle Thirds Set, hereinafter referred to simply as the Thirds set. The Thirds set is a special case of Smith-Volterra-Cantor sets in general, which are addressed in section 2. A typical treatment of the subject of Thirds sets and Smith-Volterra-Cantor sets is given by Vallin.[13]

1.1 Definitions of a Cantor middle thirds set

A typical informal description of a Cantor Middle Thirds set is as follows:

Definition 1.1.
Given the closed real interval $[0,1]$, perform the following recursive process:

> The first iteration is the removal of the central open middle third of the interval $[0,1]$. For all subsequent iterations, remove the central open middle third of each interval in the set of all numbers between 0 and 1 (including 0 and 1) that is the result of the previous iteration. The Cantor Middle Thirds set is the set that remains from infinitely many repetitions of this recursion.

Another definition which is commonly used is given by:[4],[Ch.5]

Definition 1.2.

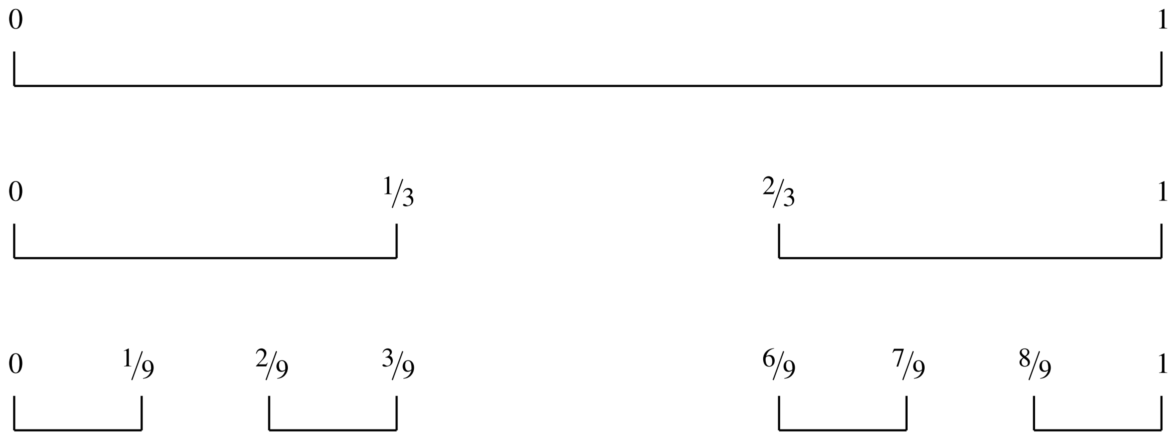
> T_0 is the interval $[0,1]$,
> T_1 is the interval $[0,1]$ less the interval $[1/3, 2/3]$,
> T_2 is the interval $[0,1]$ less the intervals $[1/9, 2/9]$, $[1/3, 2/3]$ and $[7/9, 8/9]$,
> and so on. The Thirds set is then:
>
> $$\bigcap_{n=1}^{\infty} T_n \qquad \text{where } T_n \text{ is the set of all intervals remaining after the } n^{\text{th}} \text{ iteration.}$$

A more detailed formal definition, but which is based on the union of removed open intervals, is given by:[10]

Definition 1.3.

$$[0,1]-\bigcup_{n=1}^{\infty}\bigcup_{k=0}^{3^{n-1}-1}\left(\frac{3k+1}{3^n},\frac{3k+2}{3^n}\right)$$

In terms of the iterative removal of the middle third of each interval, the first two iterations are illustrated diagrammatically below:



Following the above definition 1.3, we can formally define in first-order terms the set that consists of all the removed intervals after the n^{th} iteration:[10]

Definition 1.4. For $r \in \mathbb{R}, 0 \leq r \leq 1, n \in \mathbb{N}$

$$r \in U_n \iff \left\{ \exists m \in \mathbb{N}, 0 < m \leq n, \exists k \in \mathbb{N},\ 0 \leq k \leq (3^{m-1}-1)\ \wedge\ \frac{3k+1}{3^m} < r < \frac{3k+2}{3^m} \right\}$$

and the complementary set of remaining intervals after the n^{th} iteration is:

Definition 1.5. For $r \in \mathbb{R}, 0 \leq r \leq 1, n \in \mathbb{N}$

$$r \in T_n \iff \neg \left\{ \exists m \in \mathbb{N}, 0 < m \leq n, \exists k \in \mathbb{N},\ 0 \leq k \leq (3^{m-1}-1)\ \wedge\ \frac{3k+1}{3^m} < r < \frac{3k+2}{3^m} \right\}$$

Each set U_n corresponds to the set of points removed after the n^{th} iteration, while each set T_n corresponds to the set of points remaining after the n^{th} iteration. We can observe that each set U_n consists of the union of open non-degenerate intervals, and that each complementary set T_n consists of the union of closed non-degenerate intervals.

The Thirds set is conventionally considered to be the set that is the result of infinitely many such iterations. It is also the case that, conventionally, the elements of the Thirds set are considered to be defined as follows:

The elements of the Thirds set are defined in terms of ternary expansions, so that the numbers are represented in terms of finite or infinite expansions using only the digits $0, 1$ and 2. As with an expansion in any base, some numbers will have two different ternary expansions, for example, the ternary expansion $0.02222\ldots$ has the same numerical value as the ternary expansion 0.1 and the ternary expansion $0.12222\ldots$ has the same numerical value as the ternary expansion 0.2. This has to be taken into account in any definition involving ternary expansions.

In terms of ternary expansions, a number is said to be an element of the Thirds set provided there exists a ternary expansion of that number that does not include the digit 1 anywhere in the expansion (i.e., if there are two ternary expansions and provided that only one expansion includes the digit 1, then the number is in the Thirds set). A number is not a member of the Thirds set if it has a singular ternary expansion that has a 1 somewhere in the expansion, or if it has dual ternary expansions where both have a 1 somewhere in the expansion. For example: $1/3$ has two ternary expansions, 0.1 and 0.0222... and is in the Thirds set;. $2/3$ has two ternary expansions, 0.2 and 0.1222..., and is in the Thirds set; $4/9$ has two ternary expansions, 0.11 and 0.10222..., and is not in the Thirds set.

This informal description of the elements of the Thirds set can be stated formally in first-order terms as:[10]

Definition 1.6. For $r \in \mathbb{R}, 0 \le r \le 1$

$$r \notin \textit{Thirds set} \iff \left\{ \exists m \in \mathbb{N}, \exists k \in \mathbb{N},\ 0 \le k \le (3^{m-1} - 1) \ \wedge\ \frac{3k+1}{3^m} < r < \frac{3k+2}{3^m} \right\}$$

Definition 1.7. For $r \in \mathbb{R}, 0 \le r \le 1$

$$r \in \textit{Thirds set} \iff \neg \left\{ \exists m \in \mathbb{N}, \exists k \in \mathbb{N},\ 0 \le k \le (3^{m-1} - 1) \ \wedge\ \frac{3k+1}{3^m} < r < \frac{3k+2}{3^m} \right\}$$

Assuming the excluded middle, every real number between 0 and 1 must belong to either the Thirds set or its complement. In the Thirds set that is given by the above definition, there cannot be any non-degenerate interval that consists entirely of numbers for which there is a ternary expansion that does not include any digit as 1, since given any two such numbers, there are infinitely many numbers between two such numbers for which there is a ternary expansion that includes the digit 1. Hence the set so defined consists of isolated points. From this definition it can be readily demonstrated that this set is not denumerable.[1]–[3], [5], [6], [8], [11], [12] It is commonly observed that this is a counter-intuitive result, since on the one hand, the iterations are defined to be denumerable, and the intervals that result from the iterations are also defined to be denumerable, yet it is asserted that such iterations define a set that consists of isolated points that are not denumerable. This is a paradoxical anomaly for which no satisfactory explanation has been presented. But it should matter that expositions that yield paradoxical explanations are recognized as robust, whereas justification of an anomaly means that every aspect should be subjected to rigorously logical analysis.

Conventionally, the Thirds set is considered to be the result of infinitely many iterations of the removal of middle thirds. The justification for this conclusion is given below, where each x can be any of the digits 0, 1, or 2, except where specified:

1.1.a) For $n = 1$, any number with an expansion of the form: $0.1xxxx\ldots$, where not all $x = 0$ and not all $x = 2$, is defined by the definitions 1.5 and 1.4 to be in the set U_1, and not in the set T_1; hence such numbers are not in the Thirds set.

1.1.b) For $n = 2$, any number with an expansion of the form: $0.01xxxx\ldots$ or $0.21xxxx\ldots$, where not all $x = 0$ and not all $x = 2$, is defined by the definitions 1.5 and 1.4 to be in the set U_2, and not in the set T_2; hence such numbers are not in the Thirds set.

1.1.c) For $n = 3$, any number with an expansion of the form: $0.001xxxx\ldots$, $0.021xxxx\ldots$, $0.201xxxx\ldots$ or $0.221xxxx\ldots$, where not all $x = 0$ and not all $x = 2$, is defined by the definitions 1.5 and 1.4 to be in the set U_3, and not in the set T_3; hence such numbers are not in the Thirds set.

1.1.d) In general, any number with an expansion that has a 1 at the n^{th} digit of the ternary expansion is defined by the definitions 1.5 and 1.4 to be in the set U_n, and not in the set T_n, provided that all subsequent digits of the expansion are neither all 0 nor all 2. Alternatively, this can be stated as: Taking the ternary expansion that ends in an infinitely repeating series of 2s rather than the expansion that is finite and ends with the digit 1 or 2, then any number with an expansion that has a 1 at the n^{th} digit of the ternary expansion is defined by the definitions 1.5 and 1.4 to be in the set U_n, and not in the set T_n. Hence, if there are two ternary expansions and provided that only one expansion includes the digit 1, then the number is in the Thirds set.

Conventional descriptions[1]–[3], [5], [6], [8], [11], [12] of the Thirds set assume that a definition of the Thirds set such as 1.7 is equivalent to the result of infinitely many iterative removals of middle third closed intervals. It will be demonstrated in the remainder of this paper that this is an unsustainable assumption.

1.2 A set defined in terms of closed intervals

In order to obtain a better understanding of the situation, it is instructive to analyze sets whose definition is similar to the notion of an iterative removal of *closed* intervals rather than open intervals; hence the endpoints, rather than remaining, are also removed. Informally this gives:

> Remove the central closed middle third of each interval in the set of all numbers between 0 and 1 (excluding 0 and 1) that is the result of the previous iteration.

Formally, we have definitions similar to those of 1.4 and 1.5:

Definition 1.8. For $r \in \mathbb{R}, 0 \leq r \leq 1, n \in \mathbb{N}$

$$r \in U_n^{\text{Cl}} \iff \left\{ \exists m \in \mathbb{N}, 0 < m \leq n, \exists k \in \mathbb{N},\ 0 \leq k \leq (3^{m-1} - 1)\ \wedge\ \frac{3k+1}{3^m} \leq r \leq \frac{3k+2}{3^m} \right\}$$

Definition 1.9. For $r \in \mathbb{R}, 0 \leq r \leq 1, n \in \mathbb{N}$

$$r \in T_n^{\text{Cl}} \iff \neg \left\{ \exists m \in \mathbb{N}, 0 < m \leq n, \exists k \in \mathbb{N},\ 0 \leq k \leq (3^{m-1} - 1)\ \wedge\ \frac{3k+1}{3^m} \leq r \leq \frac{3k+2}{3^m} \right\}$$

T_n^{Cl} and U_n^{Cl} are complementary sets. We can observe that each T_n^{Cl} consists of the union of open non-degenerate intervals, and that each U_n^{Cl} consists of the union of closed non-degenerate intervals. Each set T_n^{Cl} can be said to correspond to the set of points remaining after the n^{th} iteration, while the set U_n^{Cl} can be said to correspond to the set of points removed after the n^{th} iteration.

It is trivial that all points of the form $\frac{3k+1}{3^m}$ and $\frac{3k+2}{3^m}$ are in the set U_n^{Cl}, and not in the set T_n^{Cl}. Furthermore, there is no lower limit to the size of the intervals defined by each such pair of points, although the intervals cannot degenerate altogether to single points; that is, there is no m and k such that the interval $\frac{3k+1}{3^m}, \frac{3k+2}{3^m}$ is a single point. We can refer to the intervals in terms of ternary expansions; in the following, each x can be any of the digits 0, 1, or 2:

1.2.a) For $n = 1$, any number that has an expansion of the form: $0.1xxxx\ldots$ is defined by the definitions 1.8 and 1.9 to be in the set U_1^{Cl}, and not in the set T_1^{Cl}.

1.2.b) For $n = 2$, any number that has an expansion of the form: $0.01xxxx\ldots$ or $0.21xxxx\ldots$ is defined by the definitions 1.8 and 1.9 to be in the set U_2^{Cl}, and not in the set T_2^{Cl}.

1.2.c) For $n = 3$, any number that has an expansion of the form: $0.001xxxx\ldots$, $0.021xxxx\ldots$, $0.201xxxx\ldots$ or $0.221xxxx\ldots$ is defined by the definitions 1.8 and 1.9 to be in the set U_3^{Cl}, and not in the set T_3^{Cl}.

1.2.d) In general, for a given n, any number that has an expansion that has a 1 at the n^{th} digit of the ternary expansion is defined by the definitions 1.8 and 1.9 to be in the set U_n^{Cl}, and not in the set T_n^{Cl}.

That is, for any given n, any number that has a ternary expansion with a 1 at the n^{th} digit of the expansion is defined to be in U_n^{Cl}. Note that this means that for numbers that have a dual ternary expansion, if one of the two expansions includes the digit 1 at the n^{th} digit, the number is defined to be in the set U_n^{Cl}. So we have a hierarchy of sets U_n^{Cl}, but we note that none of these sets is such that it contains every number that has a digit 1 somewhere in their ternary expansion, since each n is finite, whereas an expansion may be limitless.

1.3 Fixed ratio dividing points

Given the notion of iterative removal of intervals that can be associated with the above definitions, we can determine, in general, the numerical values of points that avoid removal by any iteration of the process. We observe that the numerical value of such points is such that they must divide every interval that they occur within, in precisely the same ratio. This means that if such a point remains after one iteration, it must remain after any subsequent iterations, since in terms of ratios, each iteration performs precisely the same process on any remaining interval. The existence of such points are determined by considering if there are any points which divide a given interval in a certain ratio, and where a remaining interval given by the subsequent iteration is such that:

(i) the point is within that interval and

(ii) the point divides that interval by precisely the same ratio.

There are two ways in which this can occur; given such a ratio, either the division is such that:

$\dfrac{\text{left-side division of the interval}}{\text{right-side division of the interval}} = \dfrac{a}{b}$ for all iterations, or

for every alternate iteration, the ratio is reversed, i.e.,

for $n = 1, 3, 5, \ldots$, $\dfrac{\text{left-side division of the interval}}{\text{right-side division of the interval}} = \dfrac{a}{b}$

for $n = 2, 4, 6, \ldots$, $\dfrac{\text{right-side division of the interval}}{\text{left-side division of the interval}} = \dfrac{a}{b}$

Since all subsequent iterations are self-similar, such a point will also divide all subsequent intervals which it is within by precisely the same ratio. To find such points, we suppose that there is such a point x in an open interval $(0,q)$, and which is such a point in the subsequent left-side open interval $\left(0,\frac{q}{3}\right)$. Then we could have either:

(i) if the ratio does not alternate at each iteration:

$$\frac{q-x}{q} = \frac{\frac{q}{3}-x}{\frac{q}{3}}$$

(ii) or if the ratio alternates direction at each iteration:

$$\frac{q-x}{q} = \frac{x-0}{\frac{q}{3}}$$

Simplifying, the first gives the value $x = 0$, which is trivially an endpoint, while the second gives $x = q/4$ i.e., for $q = 1$ this gives the point $x = 1/4$. By symmetry, or by a similar analysis, for the subsequent right-side interval $(\frac{2}{3}, 1)$, the relevant points are $x = 1$, again trivially an endpoint, and $x = 3/4$.

As noted above, all subsequent iterations are self-similar, so that each iteration will also produce further such points that are such so as to always divide every subsequent interval in precisely the same ratio. Since that is the case, there cannot be any iteration that removes such points. It will also be noted that there is a new fixed ratio dividing point for each new remaining interval produced at each iteration, see the diagram below.

It will be noted that the set of fixed ratio dividing points is a set of points that is defined in terms of a reference to the endpoints $\left[\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\right]$. For convenience, these fixed ratio dividing points are henceforth termed FRD points. In general such points are *included* among the values:

$$\frac{3k+1-\frac{1}{4}}{3^m} \text{ and } \frac{3k+2+\frac{1}{4}}{3^m}, \text{ where } 0 \leq k \leq (3^{m-1}-1).$$

but note that these formulas also include points within intervals removed by previous iterations.

These FRD points are denumerable. They are all rational numbers since they are generated by defining a ratio of the difference of two rational numbers; they all have a non-terminating ternary expansion that consists only of the digits 0 and 2, and from some point onwards in the sequence of digits consists of an infinitely repeating sequence of digits.

1.4 Infinitely many closed intervals

We now define a set U^{Cl} to be a subset of the interval $(0,1)$ which only contains numbers that have a ternary expansion with a 1 somewhere in their ternary expansions; hence the set U^{Cl} must include all numbers that have dual ternary expansions. The complementary set T^{Cl} is the set consists of numbers that have only one ternary expansion and which consists only of the digits 0 and 2. Following on from the definitions 1.8 and 1.9 we can represent this formally by:

Definition 1.10. For $r \in \mathbb{R}, 0 \leq r \leq 1$

$$r \in U^{\text{Cl}} \iff \left\{\exists m \in \mathbb{N}, \exists k \in \mathbb{N},\ 0 \leq k \leq (3^{m-1}-1) \ \bigwedge \ \frac{3k+1}{3^m} \leq r \leq \frac{3k+2}{3^m}\right\}$$

Definition 1.11. For $r \in \mathbb{R}, 0 \leq r \leq 1$

$$r \in T^{\text{Cl}} \iff \neg\left\{\exists m \in \mathbb{N}, \exists k \in \mathbb{N},\ 0 \leq k \leq (3^{m-1}-1) \ \bigwedge \ \frac{3k+1}{3^m} \leq r \leq \frac{3k+2}{3^m}\right\}$$

It will be noted that while each T_n^{Cl} given by the definition 1.9 consists of the union of non-degenerate open intervals, the points of the set T^{Cl} are isolated points[1]. Furthermore, the set T^{Cl} is not denumerable. This is easily demonstrated to be the case, as follows.

Since the set T^{Cl} consists of all numbers that have a single ternary expansion that consists only of the digits 0 and 2. We can map the elements of T^{Cl} to numbers in the interval $(0,1)$ by replacing each digit 2 by a digit 1, which gives a one-to-one correspondence, in terms of binary expansions, of all points in the set T^{Cl} to a subset S of numbers in the interval $(0,1)$. All numbers with dual ternary expansions are rationals that have a finite ternary expansion, and so have the expansions of the form $0.xxx\ldots222\ldots$ or $0.xxx\ldots2$ where x is 0 or 2.[2] This means that the only numbers that are not included in the binary expansions of $(0,1)$ by the correspondence from T^{Cl} to S are numbers of the form $0.xxx\ldots111\ldots$ or $0.xxx\ldots1$, where x is 0 or 1; these constitute a denumerable set of rationals. Hence it follows that the set S of binary expansions is not denumerable, and that the set T^{Cl} is also not denumerable.[3]

As noted in section 1.3, the FRD points given by the repeated iteration are numbers that are all rational, and all have a non-terminating ternary expansion that consists only of the digits 0 and 2, which after some digit consists of an infinitely repeating sequence of digits. However, the set T^{Cl} as given in the definition 1.11 consists of *all* numbers that have a single ternary expansion and which consists only of the digits 0 and 2; this includes expansions that do not have an infinitely repeating sequence of digits.

This demonstrates that there is a subtle difference in the formal definition of sets such as by the definition 1.11 and the intuitive notion of infinitely many iterations of the union of intervals. The difference reflects the fact that there cannot be an actual transition from a finite number of iterations to infinitely many such iterations. The notion of infinitely many iterations of the union of intervals assumes that there can be such a transition, whereas the formal definitions of 1.10 and 1.11 require no such assumption.

It will be seen that the definition of the set U^{Cl} in 1.10 is a set that consists of infinitely many isolated open intervals (whose endpoints are the points of the set T^{Cl}), whereas every iteration as defined by 1.2.d results in every U_n^{Cl} being a set that consists of infinitely many closed intervals; there cannot be any iteration where the intervals change from being closed intervals to open intervals.

This also explains why it is the case that the set T^{Cl} consists of isolated points, whereas every iteration as defined by 1.2.d results in every T_n^{Cl} being the union of open non-degenerate intervals. Furthermore, it explains why the definition of a set based on denumerable repeated iterations only results in a denumerable set of rational FRD points, whereas the set that is defined only in terms of infinitely many intervals results in a non-denumerable set of points that includes irrational points. If it were the case that the Thirds set is the result of infinitely many iterations, then by the same reasoning, since every FRD point is associated with a specific iteration that corresponds to a specific natural number, then the only points in the set T^c defined by 1.11 would be the rational FRD points.

It can be seen that while the intervals referenced by the definition of U^{Cl} constitute a denumer-

[1] The points must be isolated (as is the case for the Thirds set). This can be shown by the supposition that there exists an interval of such points of T^{Cl}. In *any* interval there must be some number with a digit 1 somewhere in its ternary expansion and which must be in that interval. Hence there cannot be any interval thus supposed.

[2] These are the points $\frac{3k+1}{3^m}$ and $\frac{3k+2}{3^m}$ in the definitions 1.10 and 1.11.

[3] See also "On the Density of Linear Sets of Points" by W. H. Young[14] re this aspect.

able set of intervals, the set that is defined by the definition consists of intervals whose endpoints are non-denumerable. The complementary set T^{Cl} is not a denumerable set of points, but the reason for this is that there is no lower limit to the size of the intervals defined by the endpoints $\frac{3k+1}{3^m}$ and $\frac{3k+2}{3^m}$. This means that the points of the set T^{Cl} are not defined in terms of any single such interval, nor in terms of any finite union of such intervals, and hence there can be no definition that enumerates the points of the set T^{Cl}.

In summary, this explanation removes all the paradoxes associated with the conventional accounts of the Thirds set, and shows that the paradoxes arise because of the intuitive assumption that the notion of infinitely many iterations of the removal of middle thirds defines the sets T^{Cl} and U^{Cl}. The same assumption has also been applied to the Thirds set.

1.5 A set defined in terms of open intervals - the Thirds set

Having dealt with the removal of closed intervals, we repeat here for convenience the definitions for the iterative removal of open intervals:

(Definition 1.4) For $r \in \mathbb{R}, 0 \leq r \leq 1, n \in \mathbb{N}$

$$r \in U_n \iff \left\{ \exists m \in \mathbb{N}, 0 < m \leq n, \exists k \in \mathbb{N},\ 0 \leq k \leq (3^{m-1} - 1)\ \wedge\ \frac{3k+1}{3^m} < r < \frac{3k+2}{3^m} \right\}$$

(Definition 1.5) For $r \in \mathbb{R}, 0 \leq r \leq 1, n \in \mathbb{N}$

$$r \in T_n \iff \neg \left\{ \exists m \in \mathbb{N}, 0 < m \leq n, \exists k \in \mathbb{N},\ 0 \leq k \leq (3^{m-1} - 1)\ \wedge\ \frac{3k+1}{3^m} < r < \frac{3k+2}{3^m} \right\}$$

The set defined by infinitely many repeated iterations of intervals according to the definition 1.5 of T_n consists of the FRD points indicated above together with the endpoints of the intervals. Hence it consists of numbers that are: rational and have a non-terminating ternary expansion that consists only of the digits 0 and 2 (including numbers that have a dual ternary expansion where one expansion consists only of the digits 0 and 2). and hence this set is an denumerable set.

On the other hand the Thirds set that is conventionally defined in terms such as:

$$r \in \textit{Thirds set} \iff \neg \left\{ \exists m \in \mathbb{N}, \exists k \in \mathbb{N},\ 0 \leq k \leq (3^{m-1} - 1)\ \wedge\ \frac{3k+1}{3^m} < r < \frac{3k+2}{3^m} \right\}$$

as in definition 1.7, and which is a definition of the complementary set of the union of closed intervals, consists of:

(i) rational numbers that have a non-terminating ternary expansion that consists only of the digits 0 and 2 (including numbers that have a dual ternary expansion where one expansion consists only of the digits 0 and 2), and

(ii) irrational numbers that have a ternary expansion that consists only of the digits 0 and 2 (including numbers that have a dual ternary expansion where one expansion consists only of the digits 0 and 2).

and hence this set is a non-denumerable set.

While all sets T_n consist of closed non-degenerate intervals, the Thirds set consists of isolated points (closed degenerate intervals). Furthermore, while all sets U_n consist of open (non-degenerate) intervals whose endpoints are all rational, the complementary set to the Thirds set consists of isolated open intervals whose endpoints must be the points of the Thirds set, and hence some such points must be irrational.

It is said that the limiting sum of the intervals that are removed by the denumerable iterative removal of middle thirds is 1, and hence the measure of the complementary set to the Thirds set cannot be less than 1; neither can it be more than 1, hence the measure of the complementary set is taken to be 1, and from this, the measure of the Thirds set must be zero. However, this method of derivation of measure cannot be applied to the general case of Smith-Volterra-Cantor sets, as will be demonstrated in Section 2.1.

2 Smith-Volterra-Cantor sets

The Thirds set is a special case of the type of set that is commonly referred to as a Smith-Volterra-Cantor[4] set[9] In the same way as for the Thirds set, it is conventionally informally said that a Smith-Volterra-Cantor set is a set that results from infinitely many recursive iterations.

Smith-Volterra-Cantor sets are based on the notion that for finitely many iterations, at the n^{th} iteration an open interval of actual (as opposed to relative) length $1/p^n$ is removed from the middle of every interval remaining from the previous iteration. For the Cantor Thirds set, the corresponding value of p is 3. If $p < 3$, the iterative process must terminate, as for some finite n the n^{th} iteration would require removal of a greater length than that remaining (this is shown below). Hence for non-terminating iterations, $p \geq 3$ must apply.[5] It can be observed that for the Thirds set, where $p = 3$, that it is also the case that a *relative* length of $1/p$ of the length of every remaining interval is removed from the middle of that interval, but it must be noted that this observation applies only to the Thirds set where $p = 3$, and it does not apply for any other value of p.

Formally, we can define the complement of the Smith-Volterra-Cantor set as:

Definition 2.1.

$$r \in SVC^{\complement} \Leftrightarrow \left\{ \exists n, k \in \mathbb{N}, n > 0, 1 \leq k \leq 2^{n-1} \;\;\wedge\;\; A(n) < r < B(n) \right\}$$

and the Smith-Volterra-Cantor set as:

Definition 2.2.

$$r \in SVC \Leftrightarrow \neg \left\{ \exists n, k \in \mathbb{N}, n > 0, 1 \leq k \leq 2^{n-1} \;\;\wedge\;\; A(n) < r < B(n) \right\}$$

where the points $A(n)$ and $B(n)$ are definable in terms of $1/p^n$ (see Appendix 3).

[4] These sets are a subset of what are termed fat Cantor sets.[4],[Ch.5]

[5] Generally it is assumed that for Smith-Volterra-Cantor sets p is a natural number, but this does not have to be the case.

A commonly cited example of a Smith-Volterra-Cantor set is when $p=4$, where, in terms of iterations, at the first iteration, an actual length of $1/4$ units is removed from the middle of the initial interval of unit length. For the second iteration, an actual length of $1/16$ units is removed from the middle of each interval that is left from the previous iteration. For the third iteration, an actual length of $1/64$ units is removed from the middle of each interval that is left from the previous iteration. And so on.

In the same way as for the Thirds Set, the Smith-Volterra-Cantor set as defined by 2.2 consists entirely of isolated points, since it cannot include any non-degenerate intervals.

2.1 The measure of a Smith-Volterra-Cantor set

The conventional analysis of the measure of a Smith-Volterra-Cantor set is as follows:

> Considering the general case, there are 2^{n-1} remaining intervals before the n^{th} iteration occurs. Since at the n^{th} iteration, an actual length of $1/p^n$ is removed from each interval, then the total amount removed by the n^{th} iteration is:
>
> $$\frac{2^{n-1}}{p^n} = \frac{1}{2}\left(\frac{2}{p}\right)^n$$
>
> Therefore, after the n^{th} iteration, the total length that has been removed is:
>
> $$\frac{1}{2}\left\{\frac{2}{p} + \left(\frac{2}{p}\right)^2 + \left(\frac{2}{p}\right)^3 + \ldots + \left(\frac{2}{p}\right)^n\right\}$$
>
> Since $\frac{2}{p} + \left(\frac{2}{p}\right)^2 + \left(\frac{2}{p}\right)^3 + \ldots + \left(\frac{2}{p}\right)^n$ is a standard geometric series, the ***limiting*** value of the series as n increases is:
>
> $$\frac{\frac{2}{p}}{1-\frac{2}{p}} = \frac{2}{p-2}$$
>
> provided $p > 2$; since for non-terminating iterations $p \geq 3$, this condition is satisfied. Hence the ***limiting*** value of the total length removed, according to this method of calculation, is half of this, which is $\frac{1}{p-2}$.[6]
>
> According to this method of calculation, for $p=3$, the limiting value of the total length removed is 1, for $p=4$ the limiting value of the total length removed is $1/2$, for $p=5$ the limiting value of the total length removed is $2/3$, and so on. According to this method of calculation, by choosing an appropriate real number value for p, any value between 0 and 1 can be given for the limiting value of the total length removed. Furthermore, for all $p \geq 3$ the limiting length of the size of a removed interval as n increases is zero, according to this method of calculation, since the actual size of a removed interval is $1/p^n$.

[6] From this it follows that for $2 < p < 3$, since the value of $\frac{1}{p-2}$ is greater than 1, the iteration must terminate at some finite n.

For $p = 4$, the conventional reading is that the recursive iterations define a set where the ***limit*** of the sum of the lengths that have been removed is $1/2$. Since the limit of the sum of the intervals removed is $1/2$, then, according to this method of calculation, the total length of what remains in the Smith-Volterra-Cantor set for $p = 4$ must be $1 - 1/2$, which is $1/2$.

It can be noted that although most authors use the approach as indicated above, at least one author employs a different method of calculating the measure of an Smith-Volterra-Cantor set. Nelson[7] asserts that the outer measure of the Thirds set is given by the following consideration:

The set given by the n^{th} iteration, C_n, consists of 2^n intervals of length $1/3^n$, so the outer measure

$$m^*(C) \leq 2^n \cdot \frac{1}{3^n} = \left(\frac{2}{3}\right)^n$$

and the only way this can hold for every positive integer n is for $m^*(C) = 0$.

In the case of Smith-Volterra-Cantor sets in general where $p > 3$, the measure of each remaining interval at the n^{th} iteration is not given in such a simple format as $1/3^n$, but nevertheless, the same logic must apply, and the measure of any Smith-Volterra-Cantor set where $p > 3$ must also be zero, since the measure of each remaining interval at the n^{th} iteration is given by a numerical value less than 1 raised to the power of n, which approaches zero as $n \to \infty$.

Note that both the above methods of calculating measure use the notion of a limit.

2.2 Limiting Values

The difference in these two different approaches outlined above can be summarized as follows:

2.2.1 Case 1: The total measure less the limiting value of the summation of measures of removed intervals

Let A_n be the set of all endpoints of removed intervals that are defined up to and including the n^{th} iteration. This is a denumerable set and a function can readily be defined that includes the elements of this set. For the n^{th} iteration, there are

$$s = \sum_{i=1}^{n} 2^i$$

endpoints, so there are s points in the set A_n. From the recursive definition of the iterative removal of intervals (see section 3), a function $f(m)$ can be readily defined that enumerates the elements of the set A_n, where $1 \leq m \leq s$, such that $f(m)$ gives the m^{th} largest point of the set A_n. Then we can define the sum of the measures of all removed intervals up to and including the n^{th} iteration as:

$$\sum_{i=1}^{n/2} \left\{ f(2i) - f(2i-1) \right\}$$

The limiting value of the measures of all removed intervals is thus

$$\lim_{i\to\infty}\sum_{i=1}\left\{f(2i)-f(2i-1)\right\}$$

The conventional analysis of the measure makes the ***assumption*** that:

$$\lim_{n\to\infty}\sum_{n=1}\frac{1}{2}\left(\frac{2}{p}\right)^n \overset{???}{=} \lim_{i\to\infty}\sum_{i=1}\left\{f(2i)-f(2i-1)\right\} \tag{2.2.1}$$

which, for $p=4$ gives:

$$\lim_{n\to\infty}\sum_{n=1}\frac{1}{2}\left(\frac{2}{4}\right)^n = \frac{1}{2} \overset{???}{=} \lim_{i\to\infty}\sum_{i=1}\left\{f(2i)-f(2i-1)\right\}$$

so that the measure of the Smith-Volterra-Cantor set under the assumption in 2.2.1 is:

$$\frac{1}{2} \tag{2.2.2}$$

2.2.2 Case 2: The limiting value of the summation of measures of remaining intervals

We now let B_n be the set A_n together with the points 0 and 1, i.e., the set of all endpoints of removed intervals that are defined up to and including the n^{th} iteration, along with the points 0 and 1. For the n^{th} iteration, there are

$$t = 2+\sum_{i=1}^{n}2^i$$

points that include the endpoints and 0 and 1, so there are t points in the set B_n. As for the function $f(m)$ for the set A_n, a function $g(m)$ can readily be defined that enumerates the elements of this set B_n, where $1 \le m \le t$, such that $g(m)$ gives the m^{th} largest point of the set B_n. Then we can define the sum of the measures of all *remaining* intervals after the n^{th} iteration as:

$$\sum_{i=1}^{n/2}\left\{g(2i)-g(2i-1)\right\}$$

Therefore, the limiting value of the measures of all remaining intervals as $n\to\infty$ is:

$$\lim_{i\to\infty}\sum_{i=1}\left\{g(2i)-g(2i-1)\right\}$$

In the case of the sets of removed intervals, the intervals already removed in a set A_n always appear in any subsequent set A_m, where $m>n$. However, this is not the case for the sets B_n of remaining intervals, and at each iteration the size of *every* interval that remains is always smaller than on the previous iteration, and since there is no lower limit to the sizes of the remaining intervals, we have, regardless of the value of p:

$$\lim_{i\to\infty}\sum_{i=1}\left\{g(2i)-g(2i-1)\right\}=0$$

This result contradicts the previous result of (2.2.2) above.

2.3 The source of the discrepancy of measure

We have in the above a demonstration that the attempt to calculate a measure by a naive application of a limiting condition may not ensure a correct result, particularly if unwarranted assumptions are made. Furthermore, it was shown in sections 1.4 and 1.5 that the conventional assumption that the Thirds set (and in general any Smith-Volterra-Cantor set) is the result of an infinite number of recursive removals of intervals is incorrect. It follows that the naive assumption that one can apply the limiting case of the measures of removed intervals by:

$$\lim_{n\to\infty}\sum_{n=1} \;=\; \frac{1}{2}\left\{\frac{2}{p}+\left(\frac{2}{p}\right)^2+\left(\frac{2}{p}\right)^3+\ldots+\left(\frac{2}{p}\right)^n\right\}$$

(as detailed above in 2.2.1) lacks any logical foundation.

It might be thought that the reason for the discrepancy is that the measure of the Smith-Volterra-Cantor set is a measure of a non-denumerable set of isolated points, whereas, for all n, the remaining set at the n^{th} iteration is a denumerable set. However, this is not the real reason for the discrepancy; this is demonstrated below in Section 2.4, where the same discrepancy can be elicited by definitions that do not define non-denumerable sets of isolated points nor non-denumerable sets of isolated intervals (every non-degenerate interval, of course, consists of a non-denumerable set of points).

The real reason for the discrepancy is the erroneous assumption that the Smith-Volterra-Cantor set is actually the result of infinitely many iterations of the removal of intervals, whereas this is not the case, as was demonstrated in sections 1.4 and 1.5, and a similar result is demonstrated in Section 2.4 below.

The crucial point that must be addressed in any attempt to apply the notion of a limiting value of a measure is the analysis as to whether the notion of a limiting value can be sensibly applied to a given case.

2.4 Other sets defined by decreasing intervals

We now examine some other sets that can be defined from the notion of iterations of the removal of intervals where each interval is a given fraction of a previous interval.

Given any function $f(n)$ that enumerates a set A of real numbers between 0 and 1, where each such real number $r\in A=f(n)$ is enclosed by a closed interval of measure $1/(2^n\cdot m)$, where $m\in\mathbb{N}, m>1$, with r at the midpoint of the interval, then the endpoints of that interval are $f(n)-1/(2^{n+1}\cdot m)$ and $f(n)+1/(2^{n+1}\cdot m)$. Hence the set A of all points in the interval $[0,1]$ that are included by such intervals is given by the following:

Definition 2.3. For $r\in\mathbb{R}, 0\le r\le 1$

$$r\in A \iff \left\{\exists n\in\mathbb{N} \;\wedge\; \left[f(n)-\frac{1}{(2^{n+1}\cdot m)} \;\le\; r \;\le\; f(n)+\frac{1}{(2^{n+1}\cdot m)}\right]\right\}$$

So, for example, given a function $q(n)$ that enumerates all rationals between 0 and 1, and where $m=4$, we define a set B by:

Definition 2.4. For $r \in \mathbb{R}, 0 \leq r \leq 1$

$$r \in B \iff \left\{ \exists n \in \mathbb{N} \wedge \left[q(n) - \frac{1}{(2^{n+1} \cdot 4)} \;\leq r \leq\; q(n) + \frac{1}{(2^{n+1} \cdot 4)} \right] \right\}$$

Clearly, every rational in the interval $(0,1)$ must be included in the set B, since every rational is referenced by the function q, and so must be within a covering interval. It is also the case that 0 is in B; otherwise that would imply that there is a rational c such that there is an interval between 0 and c, within which there is no covering interval given by the definition of B. But that cannot be the case, since c itself must be the midpoint of a covering interval, which would imply another rational $d, 0 \leq d < c$. Similarly, 1 must also be included in B. By a similar argument, it also follows that all irrationals between 0 and 1 must also be included in the set B, and hence the definition of B defines the unit interval $[0,1]$.

We now consider another definition, where

$$f(n) = \frac{(2^{n+2} - 3)}{(2^{n+3})}, \text{ and where again } m = 4$$

Definition 2.5. For $r \in \mathbb{R}, 0 \leq r \leq 1$

$$r \in C \iff \left\{ \exists n \in \mathbb{N} \; \left[\frac{(2^{n+2} - 3)}{(2^{n+3})} - \frac{1}{(2^{n+1} \cdot 4)} \;\leq r \leq\; \frac{(2^{n+2} - 3)}{(2^{n+3})} + \frac{1}{(2^{n+1} \cdot 4)} \right] \right\}$$

Consider now the set that is defined by a series of consecutive intervals where the first interval is of measure $1/4$, with its left endpoint at 0, and where each consecutive interval has a measure of half the preceding measure, which gives:

Definition 2.6. For $r \in \mathbb{R}$

$$r \in D \iff \left\{ \exists n \in \mathbb{N} \wedge \left[\frac{1}{2} - \frac{1}{2^{n+1}} \;\leq r \leq\; \frac{1}{2} - \frac{1}{2^{n+2}} \right] \right\}$$

It can be observed that this is precisely the same definition as the previous one, i.e., set C = set D. It will also be observed that the measure of each individual enumerated interval (for any given enumeration) given in the definition of the sets B and C are identical (and similarly for sets B and D).

As was indicated above, the measure of the set B is 1, but in the case of the set D, the limiting value of the rightmost endpoints is the limit of:

$$\frac{1}{2} - \frac{1}{2^{n+2}}$$

which is $1/2$, hence there cannot be any points in D (or C) greater than $1/2$. It follows that although the measure of each individual enumerated interval, for any given enumeration, is identical in the sets B and D, the total measure of the set B is 1, while the total measure of the set D (and C) is $1/2$. The conclusion is that although we have the same method of definition, with enumerated intervals that are of identical measure for any given n, in one case the total measure is $1/2$, while in the other case, the total measure is 1.

However, according to the method of calculating measure as in commonly used for Smith-Volterra-Cantor sets (as indicated above), the total measure of each of the sets B, C, and D is identical and is $^1/_2$. The calculation is as follows:

The sum of a finite number of measures is:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^{n+1}} \quad = \quad \frac{1}{2}\left\{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^n}\right\}$$

$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^n}$ is a standard geometric series, so the ***limiting*** value of the series as n increases is:

$$\frac{1}{2}\left\{\frac{\frac{1}{2}}{1-\frac{1}{2}}\right\}=\frac{1}{2}$$

This demonstrates that the conventional method of a naive application of a limiting value of the sum of the enumerated intervals does not necessarily give the correct value of the total measure of a set defined in terms of such intervals. It also demonstrates that the anomalies of measure that ensue from such application of a limiting measure are not dependent on a definition that entails denumerability.

3 Appendix: The endpoints for the general case of iterative removal of intervals

We can give a recursive definition that will give the values of the endpoints for any iteration of the removal of the open middle section of existing intervals, for any real number value of $p \geq 3$, where the actual length of $^1/p^n$ is removed from the middle of each interval. In the following definition, the L points are the left endpoints of the remaining intervals, and the R points are the right endpoints of the remaining intervals, see the diagram below.[7]

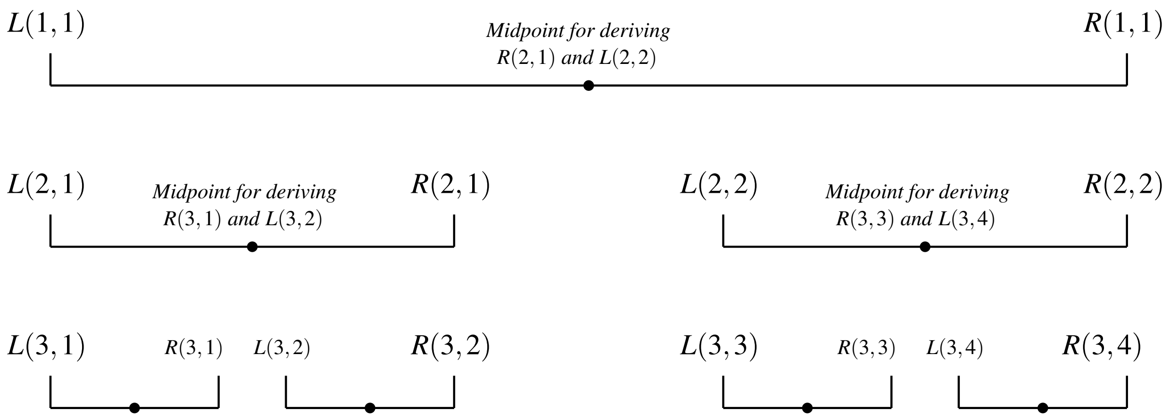
Definition A.1.

$$\text{For } \ n=0, k=0, \qquad L(n,k)=0, \ \ R(n,k)=1 \qquad \text{(A.1.a)}$$

$$\text{otherwise, for } \ n \in \mathbb{N}, n \geq 1, k \in \mathbb{N}, 1 \leq k \leq 2^{n-1}$$

$$L(n,k) = \begin{cases} 0 & \text{if } k=1 \quad \text{(A.1.b)} \\ L\left(n-1, \dfrac{k+1}{2}\right) & \text{if } k \text{ odd}, k \neq 1 \quad \text{(A.1.c)} \\ ^1\!/\!_2 \left[L\left(n-1, \dfrac{k}{2}\right) + R\left(n-1, \dfrac{k}{2}\right) + \dfrac{1}{p^{n-1}} \right] & \text{if } k \text{ even} \quad \text{(A.1.d)} \end{cases}$$

$$R(n,k) = \begin{cases} 1 & \text{if } k=2^{n-1} \quad \text{(A.1.e)} \\ R\left(n-1, \dfrac{k}{2}\right) & \text{if } k \text{ even}, k \neq 2^{n-1} \quad \text{(A.1.f)} \\ ^1\!/\!_2 \left[L\left(n-1, \dfrac{k+1}{2}\right) + R\left(n-1, \dfrac{k+1}{2}\right) - \dfrac{1}{p^{n-1}} \right] & \text{if } k \text{ odd} \quad \text{(A.1.g)} \end{cases}$$



[7] The definition is somewhat easier when done in this manner, although the endpoints could be defined as the left and right endpoints of the intervals removed at the n^{th} iteration.

In general for a given n there are $k = 2^n$ endpoints, so there are $k = 2^{n-1}$ left endpoints and $k = 2^{n-1}$ right endpoints, hence $1 \leq k \leq 2^{n-1}$. $L(n,k)$ and $R(n,k)$ indicate the left and right endpoints respectively of the remaining intervals for the n^{th} iteration. Case A.1.b ensures that the leftmost endpoint is always 0, and similarly, case A.1.e ensures that the rightmost endpoint is always 1. At each iteration, the number of endpoints doubles, and so the maximum value of k doubles at each iteration. Half of the endpoints for a given n are given by the endpoints existing before the iteration, and the other half is given by newly formed endpoints. The left unaltered endpoints always have an even k before the iteration, and an odd k after the iteration, hence the endpoint is carried over to the next value of n by case A.1.c by $\frac{k+1}{2}$; the right unaltered endpoints always have an odd k before the iteration, and an even k after the iteration, hence the endpoint is carried over to the next value of n by case A.1.f by $\frac{k}{2}$ (see the diagram below).

New endpoints for the n^{th} iteration are given by cases A.1.d and A.1.g. The midpoint of an interval remaining from the previous iteration is determined from the endpoints of that interval, i.e., $\frac{1}{2}[L(\ )+R(\ )]$, and the new endpoints are given by subtracting and adding the appropriate value to that midpoint, i.e., $\frac{1}{2p^{n-1}}$. New left endpoints always have an even k, and new right endpoints always have an odd k. The reason why $k+1$ appears in the definition for case A.1.g, while k appears in the corresponding case A.1.d is that, for a given k and a given n, the $L(n,k)$ and the $R(n,k)$ are defined in terms of the midpoint of a previous remaining interval, which itself is defined in terms of the left and right endpoints of that interval; hence the new left endpoint has a k that is 1 greater than the new right endpoint for a given previous remaining interval. The diagram above illustrates how this operates.

Having defined the endpoints, we can now define the complement of the Smith-Volterra-Cantor set for any given p.

Definition A.2.

$$r \in SVC^{\complement} \Leftrightarrow \left\{ \exists n,k \in \mathbb{N}, n > 0, 1 \leq k \leq 2^{n-1} \wedge R(n,k) < r < L(n,k+1) \right\}$$

and the Smith-Volterra-Cantor set is:

Definition A.3.

$$r \in SVC \Leftrightarrow \neg \left\{ \exists n,k \in \mathbb{N}, n > 0, 1 \leq k \leq 2^{n-1} \wedge R(n,k) < r < L(n,k+1) \right\}$$

The problems noted earlier that arises from conventional considerations of the iterative recursion, such problems are obviated when the definition is given in the above manner, which is that a real number r is defined as being an element of a set if, for a given proposition with n as a free variable, there exists an n such that the proposition is satisfied.

While it is not proved that there is an alternative definition of the Smith-Volterra-Cantor set that is not given as the complement of a set defined in terms of iterative intervals, it appears unlikely that there could be such a finite definition. It is to be noted that while the Smith-Volterra-Cantor set is a set where its elements are given by "squeezing", the same is not true of the complementary set.

However, if we examine a logical definition of the Smith-Volterra-Cantor set as given by A.3 above, we see that the Smith-Volterra-Cantor set is defined as a set of isolated points that are defined by fact that there is no lower limit to the size of the interval between $L(n,k)$ and $R(n,k)$ as $n \to \infty$. It also follows that there is no lower limit to the size that an interval between $R(n,k)$ and $L(n,k+1)$ can be as $n \to \infty$. The difference, of course, is that in the former case, each interval $L(n+1,k), R(n+1,k)$ is within some interval $L(n,j), R(n,j)$, whereas this does not apply in the latter case.

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