

The references to Peter Smith's book in this paper refer to the First Edition of 2007. An updated version of this paper with references modified to apply to the Second Edition of 2013 is available at <http://www.jamesrmeyer.com>.

**A FUNDAMENTAL FLAW IN AN INCOMPLETENESS PROOF
IN THE BOOK
“AN INTRODUCTION TO GÖDEL’S THEOREMS” BY PETER SMITH**

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Abstract

This paper examines a proof of incompleteness given by Peter Smith in a book entitled ‘An Introduction to Gödel’s Theorems’. Smith’s proof is one of a number of purported proofs of incompleteness which have the intention that the proof be simpler than Gödel’s original incompleteness proof and which are achieved by following a proof schema that is intended to be simpler than that in Gödel’s original proof. This paper shows that Smith’s proof makes erroneous assumptions regarding relations of number theory which result in contradictions and which render the proof invalid.

1 Introduction

Since Gödel published his original proof of incompleteness [3] over seventy years ago, there have been many who find his proof difficult to follow, and as a result there have been numerous attempts (see, for example Smullyan [8]) to provide proofs of incompleteness that are simpler than Gödel’s original proof. In such attempts at simpler proofs of incompleteness, there is a tendency for the authors of these simplified proofs to overlook certain fundamental logical considerations. This paper examines one such attempt at a simplified proof which is given in a book written by Peter Smith, entitled ‘An Introduction to Gödel’s Theorems’ [7].

2 Considerations of Gödel Numbering Functions

One of the key ideas behind Gödel’s incompleteness proof is that we can form a correspondence between formulas of the formal system and natural numbers, so that for every relationship between formulas of the formal system we can map that relationship precisely to relationships between natural numbers; so that if a certain relationship between formal system formulas applies, then there is corresponding relationship between natural numbers which also applies.

The system that is used in Gödel's proof [3] to map the formal system formulas to natural numbers is normally called the Gödel numbering system. Gödel numbering systems are commonly used in other proofs of incompleteness. The standard description of a Gödel numbering system proceeds as follows:

First we have a function ψ that gives a one-to-one correspondence of each formal symbol to some natural number. So we might have, for example:

Formal Symbol	Corresponding number
S	$\psi[S] \equiv 2$
0	$\psi[0] \equiv 3$
\neg	$\psi[\neg] \equiv 5$
\forall	$\psi[\forall] \equiv 7$
\exists	$\psi[\exists] \equiv 9$
$=$	$\psi[=] \equiv 11$
$($	$\psi[(] \equiv 13$
$)$	$\psi[)] \equiv 15$

For a given formal formula, this gives, by application of the ψ function, a series of number values. The second step is to apply another function to this series. This function takes each of these number values in sequence; for the n^{th} such value, the n^{th} prime number is raised to the power of that value (the value given by the ψ function), and this gives another series of number values. The final step is to take all of these values and multiply them together. This now gives a single number value. Given any formal system formula, there is a corresponding Gödel numbering for that formula, a number that is unique for that formula; the Gödel numbering preserves the uniqueness of the formulas, each formula having one corresponding number, for example:

Formal Expression	Corresponding Gödel number
0	2^3
$SSS0$	$2^2 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 11^3$
$\neg(SSSS0 = SS0)$	$2^5 \cdot 3^{13} \cdot 5^2 \cdot 7^2 \cdot 11^2 \cdot 13^2 \cdot 17^3 \cdot 19^{11} \cdot 23^2 \cdot 29^2 \cdot 31^3 \cdot 37^{15}$

And, given any number, we can also reverse the process to give the original formal system formula (note that if a number is not a Gödel number for some symbol combination of the formal system, no formal symbol combination will be given by the reverse process; in principle this is immaterial, since the Gödel numbering system could be modified so that every number is the Gödel number of some combination of formal system symbols).

It is quite evident that the Gödel numbering system is stated in a language that is a meta language to the formal system. In spite of this, there are several purported proofs of incompleteness in which there is either an implicit assumption or a direct assertion that there exist formulas of the formal system that actually include all of the information that is contained within the Gödel numbering system. Peter Smith's proof is one such example.

Before we examine Smith's proof, it is worth noting that it is generally the case that in descriptions of Gödel numbering the format of the resultant Gödel numbers given by the Gödel numbering system is ignored. In most cases the Gödel numbers are simply assumed to be natural numbers where the actual format is immaterial; whenever specific references to individual Gödel numbers are made, they are generally presented in standard decimal (base 10) format.

But if there is a distinction between natural numbers in the format in which they occur in the meta language and natural numbers in the format of the formal system, then there is similarly a distinction between a Gödel numbering function whose range is natural numbers in the format of the meta language, and a Gödel numbering function whose range is natural numbers in the format of the formal system. This can be clarified by an appropriate designation. If the function gives the Gödel numbers as natural numbers in the meta language (i.e., that are not necessarily symbol combinations of the formal system that represent natural numbers) we designate that function as $GN[n]$. If the function gives the Gödel numbers in the format of symbol combinations of the formal system that represent natural numbers, we designate that function as $FSGN[x]$. So, for example, for a $GN[n]$ function and a $FSGN[x]$ function, we might have:^a

Formal Symbol	Corresponding Gödel number as given by the $GN[n]$ function	Corresponding Gödel number as given by the $FSGN[x]$ function
S	2^2	$SSS\dots 0$
0	2^3	$SSS\dots\dots 0$
\neg	2^5	$SSS\dots\dots\dots 0$
\forall	2^7	$SSS\dots\dots\dots\dots 0$
\exists	2^9	$SSS\dots\dots\dots\dots\dots 0$
$=$	2^{11}	$SSS\dots\dots\dots\dots\dots\dots 0$
$($	2^{13}	$SSS\dots\dots\dots\dots\dots\dots\dots 0$
$)$	2^{15}	$SSS\dots\dots\dots\dots\dots\dots\dots\dots 0$

Note that the symbol combinations that represent numbers in the formal system, such as 0 , $S0$, $SS0$, $SSS0$, ... may also represent numbers in the meta language. In that case, any value given by the $FSGN[x]$ function can be a value of the meta language, but the same does not apply for the $GN[n]$ function in respect of the formal system; that is, a value given by the $GN[n]$ function is not necessarily a value that is a natural number in the format of the formal system.

^aNote that since an attempt to write out the entire series of S s would be impractical, we use here the abbreviation

3 Terminology

Before we consider the details of Peter Smith’s account of an incompleteness theorem [7], we shall first address the terminology; for the sake of clarity we will use a slightly different terminology to that in Smith’s book, as some of Smith’s terminology makes for difficult reading.

Smith refers to the ‘numeral’ of a number, by which he means the combination of symbols of the formal system that represent a natural number (see Smith’s *Section 4.3*). This is a function, and although Smith does not give it a name, we will call it *Numeral*[x], so that, for example, *Numeral*[6] indicates the symbol combination *SSSSSO* of the formal system.

Smith uses the term $\ulcorner \Phi \urcorner$ (where Φ is a variable with the domain of symbol combinations of the formal system) to represent two entirely different concepts (see Smith’s *Section 15.4*). This makes reading Smith’s account somewhat cumbersome, and for this reason, we will clearly differentiate the two concepts as follows. The two concepts represented by $\ulcorner \Phi \urcorner$ are:

- the Gödel number for Φ , where the Gödel number is the format of the meta language, not in the format of numbers of the formal system. We will use the term *GN*[Φ] instead.
- where the term $\ulcorner \Phi \urcorner$ occurs in an expression which is a representation (in the meta language) of a formal symbol combination, it represents what Smith refers to as the numeral of the Gödel number of Φ . In our terminology, this is *Numeral*[*GN*[Φ]]. It will be observed that if *FSGN*[Φ] is a Gödel numbering function that gives Gödel numbers in the format of the formal system, then *Numeral*[*GN*[Φ]] \equiv *FSGN*[Φ].

In addition, Smith uses different fonts in an attempt to distinguish symbols of the formal system and symbols of the meta language, using sans-serif fonts for symbols that are symbols of the formal system, and serif fonts for symbols of the meta language. However, he also uses sans serif fonts for expressions of the meta language that are not symbol combinations of the formal language but which represent symbol combinations of the formal language. This also makes reading his text unnecessarily cumbersome, so here all symbols of the formal system will be highlighted in gray, as in this example:

$$\exists y(y = \text{SSSSO})$$

It follows that any symbol (or combination of symbols) that is not so highlighted is not a symbol of the formal system; but it can represent (in the meta language) a symbol combination of the formal system.

Furthermore, for clarity, in this paper square brackets [] will be used to indicate brackets in the meta language, while round brackets () will be reserved to indicate brackets of the formal system.

One other term used that is slightly different to Smith’s terminology is the term *Gdl*^{*FS*}[x, y]. Given that *Gdl*[m, n] is a relation in the meta-language, Smith uses the non-italicized *Gdl*[x, y] to represent the symbol combination of the formal language that expresses this relation *Gdl*[m, n] in the formal language, whereas we will use *Gdl*^{*FS*}[x, y] to represent this concept (this is explained further on page 7).

4 Definitions used in Smith's proof

We now proceed to the examination of Smith's proof. Smith defines:

$$n = GN[\Phi] \quad (4.1)$$

where n is a variable whose domain is natural numbers, and Φ is a variable whose domain is symbol combinations of the formal system.

Theorem 15.2 of Smith's *Section 15.5* introduces a function $diag[n]$ which Smith defines as:

$$diag[n] = GN[\exists y(y = \ulcorner \Phi \urcorner \wedge \Phi)]$$

where ' $\exists y(y = \ulcorner \Phi \urcorner$ ', ' \wedge ', and ' \urcorner ' are symbols of the formal system. Since that part of the expression that is ' $\exists y(y = \ulcorner \Phi \urcorner \wedge \Phi)$ ' is intended to signify a symbol combination of the formal system, then, using the unambiguous terminology given above, we have:

$$\begin{aligned} diag[n] &= GN[\exists y(y = \text{Numeral}[GN[\Phi]] \wedge \Phi)] \\ &= GN[\exists y(y = \text{FSGN}[\Phi] \wedge \Phi)] \end{aligned} \quad (4.2)$$

where $n = GN[\Phi]$.

Smith asserts in his '*Proof for (R10)*' in his *Section 15.6* that $diag[n]$ is a primitive recursive number function. His proof of that assertion is as follows. In his '*Proof for (R8)*' in his *Section 15.6*, he introduces a primitive recursive number function which is denoted by $*$ and which is defined as:

$$\begin{aligned} m * n &= [\exists x \leq B_{m,n}][[\forall i < len[m]]\{exp[x, i] = exp[m, i]\} \\ &\quad \wedge [\forall i \leq len[n]]\{exp[x, i + len[m]] = exp[n, i]\}] \end{aligned} \quad (4.3)$$

where $B_{m,n}$, len , and exp are themselves primitive recursive.^b He uses this function $m * n$ to define (see '*Proof for (R9)*' in Smith's *Section 15.6*) a further primitive recursive number function $num[x]$ as:

$$\begin{aligned} num[0] &= 2^{19} \\ num[x+1] &= 2^{21} * num[x] \end{aligned} \quad (4.4)$$

Smith defines a function $f[n]$ in terms of this function $num[x]$ as:

$$f[n] = C_1 * num[n] * C_2 * n * C_3 \quad (4.5)$$

where C_1 , C_2 , and C_3 are numerical constants. Smith asserts that since $*$ and $num[n]$ are primitive recursive number functions, this function is a primitive recursive number function. Now, letting $C_1 = GN[\exists y(y = \ulcorner \]$, $C_2 = GN[\wedge]$, and $C_3 = GN[\urcorner]$ gives:

$$f[n] = GN[\exists y(y = \ulcorner \] * num[n] * GN[\wedge] * n * GN[\urcorner]) \quad (4.6)$$

^bSee Smith's *Section 11.8* for his definitions of len and exp , and his *Section 15.6* for his definition of $B_{m,n}$.

Smith asserts in his ‘*Proof for (R9)*’ in his *Section 15.6*, that since it can be seen that:^c

$$\text{num}[n] = GN[\text{Numeral}[n]] \quad (4.7)$$

then this gives that:

$$f[n] = GN[\exists y(y = \text{Numeral}[n]) * GN[\text{Numeral}[n]] * GN[\wedge] * n * GN[]] \quad (4.8)$$

and given that $n = GN[\Phi]$ (equation 4.1 above), we have:

$$\begin{aligned} f[n] &= GN[\exists y(y = GN[\Phi]) * GN[\text{Numeral}[GN[\Phi]]] * GN[\wedge] * GN[\Phi] * GN[]] \\ &= GN[\exists y(y = GN[\text{FSGN}[\Phi]]) * GN[\wedge] * GN[\Phi] * GN[]] \end{aligned} \quad (4.9)$$

and finally, by the assertion in Smith’s ‘*Proof for (R8)*’ in his *Section 15.6* that in general, where A and B are symbol combinations of the formal system,

$$GN[A] * GN[B] = GN[AB] \quad (4.10)$$

we have from equation 4.9 that:

$$f[n] = GN[\exists y(y = \text{FSGN}[\Phi] \wedge \Phi)] \quad (4.11)$$

which is the original definition of $\text{diag}[n]$ as in equation 4.2 above. Smith’s claim is that since $f[n]$ is defined as a primitive recursive number function (in equation 4.5 above), and since the function $f[n]$ is the function $\text{diag}[n]$, then $\text{diag}[n]$ must be a primitive recursive number function. Smith asserts that since that is the case, $\text{diag}[n]$ can be expressed in the formal system. Note that Smith frequently refers to the expression $\exists y(y = \text{FSGN}[\Phi] \wedge \Phi)$ as the ‘diagonalization’ of Φ .^d

5 Smith’s Incompleteness Formula

We follow Smith’s use of the diag function to elicit his proof of incompleteness in full detail, including detailed definitions of formal system formulas referred to in the proof,^e proceeding according to the outline argument given in Smith’s *Section 16.2*. In Smith’s *Section 15.2*, he defines a relation $\text{Prf}[m,n]$ such that $\text{Prf}[m,n]$ holds if and only if:

- m is the Gödel number of a symbol combination M of the formal system,
- n is the Gödel number of a symbol combination N of the formal system,
- M is the proof (in the formal system) of N .

^cWhere in the Gödel numbering system the function ψ assigns 19 as the number corresponding to the formal symbol 0, and 21 as the number corresponding to the formal symbol \mathcal{S} . Here we use the values given in *Section 15.4* and *Section 15.6* of Smith’s book, not as in his *Section 15.1*.

^dSee Smith’s *Section 15.5*.

^eMuch of this detail is omitted in Smith’s account

Smith asserts that this relation is a primitive recursive number relation (he provides a sketch of a proof in his *Section 15.3* and a more complete proof in his *Section 15.9*). He defines a further relation as follows:

$$Gdl[m,n] = Prf[m, diag[n]]$$

Smith asserts in his *Section 16.2* that since $Gdl[m,n]$ is defined in terms of Prf and $diag$, which are primitive recursive then $Gdl[m,n]$ is also a primitive recursive number relation. Smith now defines a formula $U[y]$ as:

$$U[y] = \forall x \neg Gdl^{FS}[x,y]$$

where ' $\forall x \neg$ ' is a symbol combination of the formal system, and $Gdl^{FS}[x,y]$ is a formal system formula that expresses in the formal system the relation $Gdl[m,n]$. Note that in Smith's account, in his *Section 16.2*, the italicized $Gdl[m,n]$ is a relation in the meta language, and the non-italicized $Gdl[x,y]$ is a meta language representation of the symbol combination for that relation in the formal language. Here, for clarity, we use the terms $Gdl[m,n]$ for the relation of the meta language, and $Gdl^{FS}[x,y]$ to represent the symbol combination of the formal language that expresses the relation $Gdl[m,n]$ in the formal language.

Since $U[y]$ is intended to be a representation of an actual formal symbol combination, if $Prf^{FS}[x,y]$ is the formula of the formal system that expresses in the formal system the relation $Prf[m,n]$, and if $diag^{FS}[y]$ is the formula of the formal system that expresses in the formal system the relation $diag[n]$, this gives:

$$Gdl^{FS}[x, y] = Prf^{FS}[x, diag^{FS}[y]]$$

so that:

$$U[y] = \forall x \neg Gdl^{FS}[x,y] \tag{5.1}$$

$$= \forall x \neg Prf^{FS}[x, diag^{FS}[y]] \tag{5.2}$$

Given the definition of $U[y]$, Smith defines a formula of the formal system G (which he calls the 'Gödel sentence') as:

$$G = \exists y (y = \ulcorner U \urcorner \wedge U[y])$$

Using the unambiguous terminology as indicated above (see Section 3), this gives Smith's formula G as defined above as:

$$G = \exists y (y = FSGN[U[y]] \wedge U[y]) \tag{5.3}$$

Smith asserts that upon examination of this formula G , we can see that it is true if and only if it is unprovable in the formal system. He asserts, that by the equivalence given by Gödel numbering, G is true if and only if there is no number m such that $Gdl[m, GN[U]]$. Hence he is asserting that G is true if and only if there is no number m such that m is

the Gödel number for a formal proof of the diagonalization of the formula with the Gödel number that is $GN[U]$. Smith does not give a detailed account of his assertions, but it is a straight-forward matter to do so, and we proceed as follows.

Smith asserts that the formula G is a formula of the formal system. Since G is a formula of the formal system then there must be some symbol combination of the formal system that the term $FSGN[U[y]]$ in that formula represents. We use Q to represent this symbol combination and substituting Q for $FSGN[U[y]]$, and substituting $\forall x \neg Prf^{FS}[x, diag^{FS}[y]]$ for $U[y]$ (as given by equation 5.2) gives G as:

$$\exists y(y = Q \wedge \forall x \neg Prf^{FS}[x, diag^{FS}[y]])$$

In this formula, the term $Prf^{FS}[x, diag^{FS}[y]]$ represents a formal system combination which is a relation with the free variables x and y . Now, although Smith asserts that $diag[n]$ and the formula G can be expressed in the formal system, he does not give a detailed definition of a formula of the formal system that might express $diag[n]$ or G . However, we can construct a full definition of these formulas from Smith's definitions and the argument that he presents.

First we let \diamond represent a symbol combination of the formal system that is a formula that expresses in the formal system the primitive recursive number function $*$ as defined in equation 4.3 above. And we let $num^{FS}[y]$ represent a symbol combination of the formal system that is a formula which expresses in the formal system the primitive recursive number function $num[x]$ (as defined in equation 4.4 above) and is defined by:^f

$$\begin{aligned} num^{FS}[0] &= SSS\dots 0 \\ num^{FS}[y + S0] &= SSS\dots\dots\dots 0 \diamond num^{FS}[y] \end{aligned}$$

Now, if $diag[n]$ is a function that can be expressed in the formal system, as Smith asserts, then we will have the corresponding result where $diag^{FS}[y]$ defines a symbol combination of the formal system that expresses in the formal system the function $diag[n]$, where y is a variable of the formal system, and where

$$y = FSGN[\Phi] \tag{5.4}$$

which corresponds to $n = GN[\Phi]$ in the definition of $diag[n]$ (as in equation 4.1), which gives:

$$diag^{FS}[y] = FSGN[\exists y(y =) \diamond num^{FS}[y] \diamond FSGN[\wedge] \diamond y \diamond FSGN[()]]$$

which corresponds to the definition of $diag[n]$ as in equation 4.11 above.

^fWhere the formal symbol combination $SSS\dots 0$ has the numerical value of 2^{19} and $SSS\dots\dots\dots 0$ has the numerical value of 2^{21} .

And, as for equation 4.7 above, we have that $num^{FS}[n] = FSGN[n]$, so that we have, corresponding to the equations 4.6 - 4.11 above:

$$diag^{FS}[y] = FSGN[\exists y(y =) \diamond num^{FS}[y] \diamond FSGN[\wedge] \diamond y \diamond FSGN[)]] \quad (5.5)$$

$$= FSGN[\exists y(y =) \diamond FSGN[y] \diamond FSGN[\wedge] \diamond y \diamond FSGN[)]] \quad (5.6)$$

$$= FSGN[\exists y(y =) \diamond FSGN[FSGN[\Phi]] \diamond FSGN[\wedge] \diamond FSGN[\Phi] \diamond FSGN[)]] \quad (5.7)$$

(by $y = FSGN[\Phi]$, as for equation 4.1)

$$= FSGN[\exists y(y = FSGN[\Phi] \wedge \Phi)] \quad (5.8)$$

(by the assertion $FSGN[A] \diamond FSGN[B] = FSGN[AB]$, as for 4.10)

so that the definition of $diag^{FS}[y]$ corresponds to the definition of $diag[n]$.

Now, $diag^{FS}[y]$ represents a symbol combination that is a formula of the formal system, and as we have seen from equation 5.5 above, can be defined in terms of $num^{FS}[y]$, $FSGN[\exists y(y =)]$, $FSGN[\wedge]$ and $FSGN[)]]$. The latter terms, $FSGN[\exists y(y =)]$, $FSGN[\wedge]$ and $FSGN[)]]$ are meta language terms which evaluate as constant values, and so the corresponding expressions in the formal language must also evaluate as constant values. We will use the terms $C_{\exists y(y =)}^{FS}$, C_{\wedge}^{FS} , and $C_{)]}$ to represent these specific formal symbol combinations, so that we have:

$$diag^{FS}[y] = C_{\exists y(y =)}^{FS} \diamond num^{FS}[y] \diamond C_{\wedge}^{FS} \diamond y \diamond C_{)]}$$

The formula G which is now given as:

$$\begin{aligned} G &= \exists y(y = Q \wedge \forall x \neg Prof^{FS}[x, diag^{FS}[y]]) \\ &= \exists y(y = Q \wedge \forall x \neg Prof^{FS}[x, C_{\exists y(y =)}^{FS} \diamond num^{FS}[y] \diamond C_{\wedge}^{FS} \diamond y \diamond C_{)]}) \end{aligned}$$

implies, by the rules of logic, the formal system formula:

$$\forall x \neg Prof^{FS}[x, C_{\exists y(y =)}^{FS} \diamond num^{FS}[Q] \diamond C_{\wedge}^{FS} \diamond Q \diamond C_{)]}] \quad (5.9)$$

which is obtained by the substitution of Q for the free variable y in the formula

$$\forall x \neg Prof^{FS}[x, C_{\exists y(y =)}^{FS} \diamond num^{FS}[y] \diamond C_{\wedge}^{FS} \diamond y \diamond C_{)]}]$$

So, if G is true, then the above formula 5.9 is true. From that formula, we have that there is no value of x for which

$$Prof^{FS}[x, C_{\exists y(y =)}^{FS} \diamond num^{FS}[Q] \diamond C_{\wedge}^{FS} \diamond Q \diamond C_{)]}] \quad \text{applies.}$$

Applying the correspondence given by Gödel numbering to the above formula gives us that there cannot be any proof of the formula that corresponds by Gödel numbering to the numerical value of:

$$C_{\exists y(y=)}^{FS} \diamond num^{FS}[Q] \diamond C_{\wedge}^{FS} \diamond Q \diamond C_{)}^{FS} \quad (5.10)$$

Applying the steps corresponding to equations 4.6 - 4.11 above, and since $Q = FSGN[U[y]]$, we have:

$$\begin{aligned} & C_{\exists y(y=)}^{FS} \diamond num^{FS}[Q] \diamond C_{\wedge}^{FS} \diamond Q \diamond C_{)}^{FS} \\ &= FSGN[\exists y(y=) \diamond num^{FS}[Q] \diamond FSGN[\wedge] \diamond Q \diamond FSGN[)] \\ &= FSGN[\exists y(y=) \diamond num^{FS}[Q] \diamond FSGN[\wedge] \diamond FSGN[U[y]] \diamond FSGN[)] \\ &= FSGN[\exists y(y=) \diamond FSGN[Q] \diamond FSGN[\wedge] \diamond FSGN[U[y]] \diamond FSGN[)] \\ &= FSGN[\exists y(y=) \diamond FSGN[FSGN[U[y]]] \diamond FSGN[\wedge] \\ &\quad \diamond FSGN[U[y]] \diamond FSGN[)] \\ &= FSGN[\exists y(y= FSGN[U[y]] \wedge U[y])] \end{aligned}$$

So by the correspondence given by Gödel numbering, since

$$C_{\exists y(y=)}^{FS} \diamond num^{FS}[Q] \diamond C_{\wedge}^{FS} \diamond q \diamond C_{)}^{FS} = FSGN[\exists y(y= FSGN[U[y]] \wedge U[y])]$$

then by the formula 5.9, there is not a proof of the formula

$$\exists y(y= FSGN[U[y]] \wedge U[y])$$

which is the formula G as in 5.3 above. Thus the above analysis gives the result that Smith asserts, and we note that the above result has been obtained by precisely following Smith's outline of a proof as given in his book.

6 The analysis of Smith's proof

At first glance the above analysis of Smith's argument appears to confirm Smith's outline assertions. But if we examine Smith's argument in depth, we will see that it conceals various anomalies.

The step 5.7 - 5.8 relies on the assertion that $\exists y(y=, FSGN[\Phi], \wedge, \Phi, \text{ and })$ each represent a formal symbol combination. That is a necessary assertion, since without that assertion, it cannot be asserted that

$$diag^{FS}[y] = FSGN[\exists y(y= FSGN[\Phi] \wedge \Phi)]$$

an assertion that is required for the rest of Smith's proof.

Now, since \mathbf{y} is a variable, and since $\mathbf{y} = FSGN[\Phi]$, it follows, for the formula 5.7, which is:

$$diag^{FS}[\mathbf{y}] = FSGN[\exists \mathbf{y}(\mathbf{y} = _)] \diamond FSGN[FSGN[\Phi]] \diamond FSGN[\wedge] \diamond FSGN[\Phi]$$

that on the left-hand side, \mathbf{y} is the free variable, while on the right-hand side, Φ is the free variable. There is a definite relationship between the free variable term \mathbf{y} on the left-hand side and the free variable term Φ on the right-hand side of the equation, and which is given as $\mathbf{y} = FSGN[\Phi]$ (as in 5.4 above).

But it is fundamental that *any* formal symbol combination may be substituted for Φ . This results in an irredeemable contradiction, since the expression $FSGN[\Phi]$ is itself necessarily defined as representing a formal symbol combination. We now substitute the free variables on both sides of the above equation; on the right-hand side we substitute Φ by the formal symbol combination $FSGN[\Phi]$, and on the left-hand side we substitute \mathbf{y} by the appropriate formal system numeral C (a constant whose value is given by the appropriate substitutions in the formula $\mathbf{y} = FSGN[\Phi]$), which gives the formula:

$$diag^{FS}[C] = FSGN[\exists \mathbf{y}(\mathbf{y} = _)] \diamond FSGN[FSGN[FSGN[\Phi]]] \diamond FSGN[\wedge] \diamond FSGN[FSGN[\Phi]] \diamond FSGN[_]$$

Now, since the free variables on both the left-hand side and right-hand side of the equation have been substituted, both sides of the resultant formula must evaluate as a fixed value. But that is not the case, because the right-hand side contains the term $FSGN[FSGN[\Phi]]$, the value of which does not have a singular value, but is dependent on the value of Φ .

The immediate cause of the contradiction is obvious, since the free variable \mathbf{y} on the left-hand side of the formula 5.7 is a free variable of the formal language, whereas the free variable Φ on the right-hand side is a free variable of the meta language. The same applies to the formula 5.4 ($\mathbf{y} = FSGN[\Phi]$). These anomalies demonstrate that Smith's argument has confused the formal language and the meta language.

The root cause of this confusion of language is Smith's incorrect assertion that the formula $diag[n]$, as he defines it, is a primitive recursive number function, which leads to his flawed assertion that there is an expression in the formal system that expresses that formula.

Primitive recursive number functions and relations are defined by Smith in *Section 11.2* of his book. It is clear by the definition that primitive recursive is defined for number functions and relations which have, among other properties, free variables that have the domain of natural numbers. In *Section 13.3* of his book, Smith asserts that a formal language which has variables whose domain is natural numbers and which satisfies certain other conditions can 'express' any primitive recursive number function or relation.

In Smith's '*Proof for (R8)*' in his *Section 15.6*, he defines a function, (also see equation 4.3 above):

$$m * n = [\exists x \leq B_{m,n}][[\forall i < len[m]]\{exp[x, i] = exp[m, i]\} \wedge [\forall i \leq len[n]]\{exp[x, i + len[m]] = exp[n, i]\}] \quad (6.1)$$

which is defined only in terms of variables whose domain is natural numbers. Smith asserts and proves that this is a function that satisfies his definition of a primitive recursive number function.

However, Smith then assumes that the variables m and n can be defined as any Gödel numbers; that is, he implicitly assumes that terms such as $GN[p]$ and $GN[q]$, where p and q are any symbol combinations of the formal system, may be substituted for the variables m and n to give the function:

$$\begin{aligned} m * n = & [\exists x \leq B_{GN[p], GN[q]}][[\forall i < len[GN[p]]]\{exp[x, i] = exp[GN[p], i]\}] \\ & \wedge [\forall i \leq len[GN[q]]]\{exp[x, i + len[GN[p]]] = exp[GN[q], i]\} \end{aligned} \quad (6.2)$$

so that Smith's assertion is that:

$$\begin{aligned} & [\exists x \leq B_{m, n}][[\forall i < len[m]]\{exp[x, i] = exp[m, i]\}] \\ & \quad \wedge [\forall i \leq len[n]]\{exp[x, i + len[m]] = exp[n, i]\}] \\ = & [\exists x \leq B_{GN[p], GN[q]}][[\forall i < len[GN[p]]]\{exp[x, i] = exp[GN[p], i]\}] \\ & \quad \wedge [\forall i \leq len[GN[q]]]\{exp[x, i + len[GN[p]]] = exp[GN[q], i]\} \end{aligned} \quad (6.3)$$

However, Smith also assumes that since the function 6.1 satisfies his definition of a primitive recursive number function then the function 6.2 is also necessarily a primitive recursive number function. But any assertion of equivalence/equality is an assertion that the properties of the entities for which equivalence/equality is claimed are identical within the context of that assertion. So while it may be correct that an assertion of equality of 6.1 and 6.2 is correct with regard to the property of *numerical value* in the context of a system comprising of the rules and axioms of arithmetic together with the definition of the Gödel numbering system, that assertion of equality does not apply to the property of being a primitive recursive number function.

But Smith asserts that the function 6.2 is a primitive recursive number function because the function 6.1 is a primitive recursive number function, and asserts that since the function 6.1 can be expressed by a formula of the formal system, then it must necessarily be the case that the function 6.2 can be expressed by a formula of the formal system.

This is an elementary logical error. The assumption that the function 6.2 also satisfies a rigorous definition of a primitive recursive number function has no logical foundation; and since the function 6.2 includes the free variables p and q whose domain is not natural numbers it clearly does not satisfy the definition of a primitive recursive number function. It is plainly evident that the function 6.2 cannot be expressed in the formal system, since no variable of the formal system has the domain of all the symbol combinations of the formal system, while the variables p and q in the expression 6.2 have the domain of all the symbol combinations of the formal system.

Smith is not alone in his disregard of the precise definition of number-theoretic functions and relations, and in his failure to observe that the numerical equality of two entities does not necessarily imply that those entities have precisely identical properties in all respects. There have been similar treatments in various treatises on incompleteness proofs over a considerable period, see, in particular Smullyan [8, 9], and also Boolos [1], Franzén [2], Lind [4], Most [5], and Nagel [6].

7 Further Analysis

Further analysis of Smith's assumptions may be of assistance in understanding how such errors may be avoided. If we consider Smith's assumption in the assertion of 4.7 above (which is Smith's assertion in his 'Proof for (R9)' in his Section 15.6):

$$num[n] = GN[Numeral[n]] \quad (7.1)$$

we see that the assumption is that since $num[n]$ is a primitive recursive number function then $GN[Numeral[n]]$ is also a primitive recursive number function and so it is expressible in the formal system. In this case, the free variable n on both sides of the equation does have the domain of natural numbers. However, the original definition of $num[n]$ is as a primitive recursive number function (see equation 4.4 above) with variables that have the domain of natural numbers, whereas the Gödel numbering function GN is defined with a free variable that has the domain of symbols and symbol combinations of the formal system.

Smith's proof necessarily depends on an assumption (as demonstrated in equation 5.6 above) that an equation exists for the formal system that corresponds to the above equation $num[n] = GN[Numeral[n]]$. This equation (as in equations 5.5 - 5.6) must be either:

$$num^{FS}[\mathbf{y}] = FSGN[\mathbf{y}] \quad (7.2)$$

where the variable \mathbf{y} on both sides of the equation is a variable of the formal system, or:

$$num^{FS}[\mathbf{y}] = FSGN[y] \quad (7.3)$$

where the variable \mathbf{y} on the left-hand side is a variable of the formal system, and the variable y on the right hand side is a variable of the meta-language.

Now the error in Smith's assumptions regarding primitive recursive number-theoretic expressions becomes readily apparent. Clearly equation 7.2 cannot be correct, since the variable of $FSGN$ must have the domain of all symbol combinations of the formal system, whereas \mathbf{y} has the domain only of natural numbers.

But equation 7.3 cannot be correct either. This equation must be an equation of the meta-language. This is so, since:

- a) it contains a variable y of the meta-language, and
- b) a variable of the formal language, such as \mathbf{y} cannot be an active variable in expressions of the meta language. In the meta-language the variable \mathbf{y} is a specific value, not a variable; it is one of the specific values of the domain of the free variable Φ of the Gödel numbering function $GN[\Phi]$, which is a function of the meta-language, and Φ is a variable of the meta-language.⁸

And since equation 7.3 is an equation of the meta-language, then it must follow the common rules for such equations. That means that since the left-hand side contains no variable of the meta-language, then the right-hand side should evaluate as a constant value, regardless of the value of y . But it does not evaluate as a constant value.

⁸The claim that $FSGN[\mathbf{y}]$ is intended to represent a purported expression of the formal language does not alter the above facts.

The conclusion is that the assumption generated by the Smith's incorrect assertion regarding primitive recursive number-theoretic expressions leads in either case to a logical absurdity.

Assertions such as $num^{FS}[\mathbf{y}] = FSGN[y]$ are commonly justified by the insertion of various values for \mathbf{y} and y . In this way the result is deemed to be so obvious that further analysis is unwarranted. But the use of specific instances in this way ignores the difference between meta language and formal language. The assumption is that such instances circumvent the need for an actual proof of the proposition $num^{FS}[\mathbf{y}] = FSGN[y]$. But it is an accepted principle of mathematics that a finite number of positive results cannot prove the general result for an infinite number of cases. For example, it is not accepted that Goldbach's conjecture is proven simply on the basis that it has been confirmed to apply for a large but finite quantity of natural numbers. By ignoring this principle, and by invoking specific instances as justification for the assertion that $num^{FS}[\mathbf{y}] = FSGN[y]$, the fundamental distinction between the formal language and the meta language is obfuscated.

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